EXAMPLE: QUADRATIC CONSTRAINT

$$f(x) = (Ax+b)^T(Ax+b) - c^Tx - d \le 0$$

$$A \equiv [a_1\ a_2\ \dots\ a_k]$$

$$c, x \in \mathbb{R}^k, \quad b, d \in \mathbb{R},$$

DUAL FORM

$$F(x) = F_0 + x_1 F_1 + \dots + x_k F_k$$

$$F_0 = \begin{pmatrix} 1 & b \\ b & d \end{pmatrix}, \quad F_i = \begin{pmatrix} 0 & a_i \\ a_i & c_i \end{pmatrix}$$

$$f(x) = \begin{pmatrix} 1 & b + \sum x_i a_i \\ b + \sum x_i a_i & d + \sum c_i x_i \end{pmatrix} \geq 0$$

$$\underbrace{(d + \sum c_i x_i) - (b + \sum x_i a_i)^2}_{\det(F(x))} \geq 0$$

$$\text{MIN } f_0(x)$$

$$\text{ST } f_1(x) \le 0$$

$$f_2(x) \le 0$$

$$f_L(x) \le 0$$

$$f_i(x) = (A_i x + b_i)^2 - c_i^T x - d_i$$

$$f_0(x) \le t \quad \& \quad \text{MIN } t$$

$$(A_0 x + b_0)^2 - C_0^T X - d_0 - t \le 0$$

$$\text{MIN } t$$

$$\text{S.T.} \quad \begin{pmatrix} 1 & A_0 x + b_0 \\ A_0 x + b_0 & c_0^T x + d_0 + t \end{pmatrix} \ge 0$$

$$\begin{pmatrix} 1 & A_i x + b_i \\ A_i x + b_i & c_i^T x + d_i \end{pmatrix} \ge 0 \quad i = 1 \ldots L$$

$$F_0 = \begin{pmatrix} \begin{matrix} 1 & b_0 \\ b_0 & d_0 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & b_1 \\ b_1 & d_1 \end{matrix} \; \ldots \end{pmatrix}$$

$$
F_i = \begin{pmatrix}
0 & a_{0i} & & & \\
a_{0i} & c_{0i} & & & \mathbf{0} \\
& & 0 & a_{1i} & \\
\mathbf{0} & & a_{1i} & c_{1i} & \\
& & & & \ddots
\end{pmatrix}
$$

$$
F_t = \begin{pmatrix}
0 & 0 & & \mathbf{0} \\
0 & 1 & & \\
& & 0 & \\
& \mathbf{0} & & \ddots
\end{pmatrix}
$$

CHAPTER 4 — SEMIDEFINITE PROGRAMMING

— MATOUSEK & GARTNER

4.1 LARGEST EIGENVALUE

THEOREM 4.1.1

LET $C \in \mathbb{R}^{n \times n}$ BE SYMMETRIC WITH LARGEST EIGENVALUE λ. THEN

$$\lambda = \max \{ x^T C x : x \in \mathbb{R}^n : \|x\| = 1 \}$$

→ PROOF USING DUALITY OF SDP.

GENERATING MATRICES:

$v \in \mathbb{R}^n$ IS LINEAR COMBINATION OF $e_1 \ldots e_n$

$v \in \mathbb{R}^n_+$ IS —"— OF $e_1 \ldots e_n$ WITH NON-NEGATIVE COEFFICIENTS.

IF $C \in \mathbb{R}^{n \times n}$ IS A SYMMETRIC MATRIX THEN $C = \sum \lambda_i v_i v_i^T$ WHERE

$\lambda_i \in \mathbb{R}$, $\|v_i\| = 1$ $v_i \in \mathbb{R}^n$

IF M POSITIVE SEMIDEFINITE, THEN

$M = \sum \lambda_i \, s_i s_i^T$

WHERE $\lambda_i \geq 0$, $\|s_i\| = 1$

NOT FINITELY GENERATED

LEMMA 4.1.2

$M \in \mathbb{R}^{n \times n}$ IS SYMMETRIC (POS. SEMIDEF)

IFF

$\exists s_1 \dots s_n \in S^{n-1} \quad \exists \lambda_1 \dots \lambda_n \in \mathbb{R} \quad (\lambda_i \geq 0)$

SUCH THAT $M = \sum \lambda_i \, s_i s_i^T$

PROOF

$\Leftarrow$ EASY

$\Rightarrow$ FROM DIAGONALIZATION

$$M = S D S^T = \sum S D^{(i)} S^T = \sum \lambda_i \, s_i s_i^T$$

$$D^{(i)} = \begin{pmatrix} 0 & & \\ & 0 & & 0 \\ & & \lambda_i & \\ 0 & & & 0 \end{pmatrix}$$

$\square$

THEOREM 4.11 PROOF:

WANT $\max x^T C x$

$\quad$ s.t. $\|x\| = 1$

SAME AS

$$(P)\quad \begin{cases} \max \operatorname{Tr}(C^T x x^T) \\ \text{s.t. } \operatorname{Tr}(x x^T) = 1 \end{cases}$$

SDP RELAXATION: $xx^T - X$

$$(SDP)\quad \begin{cases} \max \operatorname{Tr}(C^T X) \\ \text{s.t. } \operatorname{Tr}(X) = 1 \\ \quad X \succeq 0 \end{cases}$$

& RANK$(X) = 1$ (EXACTLY $xx^T = X$)

NO VALUE OF OPT SOLUTIONS OF (P) & (SDP) IS THE SAME.

LET X FEASIBLE $X = \sum \mu_i x_i x_i^T$

$\|x_i\| = 1$, $\mu_i \geq 0$

$\operatorname{Tr}(x_i x_i^T) = 1 \Rightarrow \sum \mu_i = 1$ SO

$\operatorname{Tr}(C^T X) = \operatorname{Tr}(C^T \sum \mu_i x_i x_i^T) =$

$\sum \mu_i \operatorname{Tr}(C^T x_i x_i^T) \leq \max \operatorname{Tr}(C^T x_i x_i^T) \leq \gamma$

SINCE x_i FEASIBLE FOR (P)

DUAL (SDP):

$$(DSDP)\begin{cases} \text{MINIMIZE } \gamma \\ \text{S.T. } \gamma I_n - C \succeq 0 \end{cases}$$

(SDP) IS STRICTLY FEASIBLE: $\left(X = \frac{1}{n} I\right)$

$\Rightarrow$ NO DUALITY GAP $\Rightarrow$

$\Rightarrow$ VALUE OF (DSDP) IS ALSO γ

LET C HAVE EIGENVALUES $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

THEN $\gamma I_n - C_n$ HAS EIGENVALUES

$\gamma - \lambda_1 \leq \dots \leq \gamma - \lambda_n$.

$\gamma I_n - C_n \succeq 0 \Rightarrow \forall i \ \gamma - \lambda_i \geq 0$

EQUALITY HOLDS IF $\gamma = \lambda_1$

$\square$