DUALITY OF SDP

$$(SDP)\begin{cases} \text{MAXIMIZE } \operatorname{Tr}(C^T X) \\ \text{SUBJECT TO } \quad A(X) = b \\ \qquad\qquad\qquad\quad X \succeq 0 \end{cases}$$

WHERE $A : SYM_n \to \mathbb{R}^m$ LINEAR OPERATOR

THEOREM:

IF (SDP) IS STRICTLY FEASIBLE AND HAS FINITE VALUE γ THEN

$$\min b^T y$$
$$\text{S.T.} \quad A^T(y) - C \succeq 0$$

IS FEASIBLE AND HAS FINITE VALUE $\beta = \gamma$.

A^T .. ADJOINT OPERATOR

(GENERALIZATION OF TRANSPOSED MATRIX

USING CLOSED CONVEX CONE DUALITY:

DEF: $K \subseteq V$ BE NONEMPTY & CLOSED.

FINITE DIM VECTOR SPACE ON $\mathbb{R}$ WITH SCALAR PRODUCT

K IS CALLED CONVEX CONE IF

1) $\forall x \in K \ \forall \lambda \in \mathbb{R}_{\geq 0} \ \ \lambda x \in K$

2) $\forall x, y \in K \ \ x + y \in K$

1) => CONE, 2 => CONVEXITY

LEMMA:

POSITIVE SEMIDEFINITE $n \times n$: $PSD_n \subseteq SYM_n$

IS A CLOSED CONVEX CONE.

PROOF:

1), 2), FROM DEFINITION, CLOSEDNESS BY COMPLEMENT BEING OPEN: $x^T M x < 0 \Rightarrow$

$\Rightarrow x^T M' x < 0$ FOR SMALL PERTURBATION. $\square$

EX $K = \{ x \in \mathbb{R}^n, x \geq 0$ IN EVERY COORDINATE $\}$ $= \mathbb{R}^n_+$

$K = \{ 0 \in \mathbb{R}^n \}$

EX V, W REAL FINITE DIMENSIONAL VECT. SPACES

$K \subseteq V$, $L \subseteq W$ CLOSED CONVEX CONES

$K \oplus L := \{(x,y) \in V \oplus W : x \in K, y \in L\}$

IS CLOSED CONE

DIRECT SUM

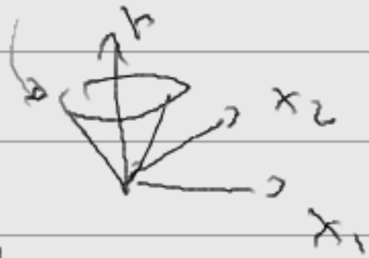
$(x,y)+(x',y') = (x+x', y+y')$

$\lambda(x,y) = (\lambda x, \lambda y)$

$\langle (x,y) | (x',y') \rangle = \langle x, x' \rangle + \langle y, y' \rangle$

EX ICE CREAM CONE:

$\nabla_n = \{(x,r) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\| \le r\}$



CLOSED

CONVEX FROM $\|x+y\| \le \|x\| + \|y\|$

∇_3

EX TOPPLED ICE CREAM CONE

$\triangleleft = \{(x,y,z) \in \mathbb{R}^3 : x \ge 0, y \ge 0, xy \ge z^2\}$

→ ALTERNATIVE $\begin{pmatrix} x & z \\ z & y \end{pmatrix}$ IS POSITIVE SEMIDEFINITE

BUT NOT THE SAME VECTOR SPACE

$$\mathbb{R}^3 \neq \mathrm{Sym}_2$$

$$\mathrm{Tr}\left(\begin{pmatrix} x & z \\ z & y \end{pmatrix}\begin{pmatrix} x' & z' \\ z' & y' \end{pmatrix}\right) = xx' + yy' + 2zz'$$

BUT

$$(x,y,z)^T(x',y',z') = xx' + yy' + zz'$$

DUAL CONES:

DEF:

$K \subseteq V$ CLOSED CONE. THEN

$$K^* = \{ y \in V : \langle y | x \rangle \geq 0 \ \forall x \in K \}$$

ex K^* IS CLOSED CONVEX CONE $\rightarrow$ FROM DEF.

Ex $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ — SELF DUAL CONE

$$0^* = V$$

LEMMA: $\triangleleft^* = \{ (x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ xy \geq z^2/4 \} \subseteq \mathbb{R}^3$

LEMMA $K \subseteq V$, $L \subseteq W$ BE CLOSED CONES
THEN $(K \oplus L)^* = K^* \oplus L^*$
CAN DO HW.

LEMMA: 3.3.1
$K \subseteq V$ CLOSED CONVEX CONE
$(K^*)^* = K$

? SEPARATION THEOREM NEEDED ?

THEOREM 3.3.2
LET $K \subseteq V$ CLOSED CONVEX CONE AND
LET $b \in V \setminus K$. THEN $\exists y \in V$ S.T.
$$\langle y | x \rangle \geq 0 \quad \forall x \in K$$
AND $\langle y | b \rangle < 0$
PROOF BY LINALGEBRA. $\square$

PROOF 3.3.1
$K \subseteq (K^*)^*$: $b \in K$ BY DEF OF K^* $\forall y \in K^*$:
: $\langle y | b \rangle = \langle b | y \rangle \Rightarrow b \in (K^*)^*$
LET $b \in K \notin V \Rightarrow \exists y$ S.T. $\langle y | x \rangle \geq 0 \quad \forall x$

AND
$\langle y | b \rangle = \langle b | y \rangle < 0$
$\Rightarrow y \in K^*$ &
& $b \notin (K^*)^*$