FARKAS LEMMA: (FOR CONES)

LEMMA: (FARKAS)

EITHER

1) $Ax = b, \; x \geq 0$ HAS SOLUTION

OR

2) $A^T y \geq 0, \; b^T y < 0$

HAS SOLUTION. BUT NOT BOTH!

REFORMULATION:

$K = \{Ax : x \in \mathbb{R}^n_+\} \subseteq V = \mathbb{R}^m$

$\rightarrow$ CLOSED CONVEX CONE

IF $Ax = b$ HAS SOLUTION $\Rightarrow b \in K$

NO SOLUTION $\Rightarrow b \in V \setminus K$

$\Rightarrow$ THM 3.32 $\Rightarrow$

$\Rightarrow \exists y : \; y^T A x \geq 0 \; \forall x \in \mathbb{R}^n_+$ & $y^T b < 0$

$\langle A^T y, x \rangle \geq 0$ $\forall x \in \mathbb{R}^n_+$ $\Downarrow$

$A^T y \in (\mathbb{R}^n_+)^* = \mathbb{R}^n_+$

(NOT BOTH 1 & 2 SOLVABLE EASILY BY 1) $y^T \cdot = y^T$

GENERALIZATION:

$$\underbrace{A}_{\text{LINEAR OPERATOR } V \to W}(x) = \underbrace{\geq}_{W} \quad x \in \underbrace{K}_{\text{SOME CONE}} \subseteq V$$

TROUBLES:

- A^T?
- $\underbrace{\{A(x),\ x \in K\}}_{\text{CONE}}$ NOT CLOSED CONE

DEF:

$A : V \longrightarrow W$ LINEAR OPERATOR.
LINEAR OPERATOR $A^T : W \to V$ IS
ADJOINT OF A IF $\forall x \in V$, $\forall y \in W$

$$\langle y \mid A(x) \rangle = \langle A^T(y) \mid x \rangle$$

NOTE:

- IF A MATRIX, ADJOINT IS A^T
- IF A^T EXISTS, UNIQUE
- IF V, W FINITE DIMENSION, A^T EXISTS

FARKAS LEMMA 1ST TRY

LET $K \subseteq V$ CONE, $b \in W$

EITHER

1) $A(x) = b$, $x \in K$ HAS SOLUTION

OR

2) $A^T(y) \in K^*$, $\langle b, y \rangle < 0$, $y \in W$

HAS SOLUTION BUT NOT BOTH.

NOT BOTH SOLVABLE TOGETHER: IF $A(x) = b$ THEN

$0 > \langle y | b \rangle = \langle y | Ax \rangle = \langle A^T y | x \rangle \geq 0$

- AT LEAST ONE HAS SOLUTION | IF $C = \{A(x), x \in K\}$ CLOSED (THM 3.3.2 NEEDS IT)

EX $K = \triangleleft\!\!D \subseteq \mathbb{R}^3 = \{x \geq 0, y \geq 0, xy \geq z\}$

LET
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \{(x, z) \in \mathbb{R}^2 : (x, y, z) \in \triangleleft\!\!D\} = 0 \cup \{x > 0\} \times \mathbb{R}$$

NOT CLOSED

FOR $b = (0, 1)$, NONE OF 1), 2) SOLVABLE.

1) CLEAR

$y = (\pm ?, -?)$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \to 0 \Rightarrow$ NO SOLUTION

2) $A^T y \in \triangleleft\!\!D^*$ SEE $\triangleleft\!\!D^*$

FARKAS LEMMA 2^{ND} TRY

LEMMA:

$K \subseteq V$ CLOSED CONVEX CONE,

$C = \{A(x) : x \in K\}$

CLOSURE OF $C = \bar{C}$ IS CLOSED CONVEX CONE

DEFINITION:

$A(x) = b,\ x \in K$ IS SUBFEASIABLE

IF $\exists (X_k)_{k \in \mathbb{N}} \subseteq K$ S.T. $\lim_{k \to \infty} A(X_k) = b$

(THEN $b \in \bar{C}$)

LEMMA (FARKAS FOR CONES)

EITHER $A(x) = b,\ x \in K$ IS SUBFEASIBLE

OR $A^T(y) \in K^*,\ \langle b, y \rangle < 0$ HAS SOLUTION

PROOF:

- $A(x) = b$ $x \in K$ IS SUBFEASABLE: $(X_k)_{k \in \mathbb{N}}$ IS SEQ

FOR y: $\langle y, b \rangle = \langle y, \lim_{k \to \infty} A(X_k) \rangle =$

$= \lim_{k \to \infty} \langle y, A(X_k) \rangle = \lim_{k \to \infty} \langle A^T(y), X_k \rangle \geq 0$

$\Rightarrow$ OR HAS NO SOLUTION. IF $A^T(y) \in K^*$

- $A(x) = b$ NOT SUBFEASABLE - FROM SEPARATION THM

CONE PROGRAMS

DEFINITION:

$K \subseteq V$, $L \subseteq W$ CLOSED CONVEX CONES,

$b \in W$, $c \in V$, $A: V \to W$ LINEAR OP.

CONE PROGRAMS:

$$(P) \begin{cases} \text{MAX} & \langle c, x \rangle \\ \text{S.T.} & b - A(x) \in L \\ & x \in K \end{cases}$$

- IF $L = \{0\}$, WE GET EQUATIONS
- FEASIBLE: $\exists x \in K$ S.T. $b - A(x) \in L$
- VALUE OF (P)

  $\sup\{\langle c, x \rangle, \ x \text{ FEASIBLE}\}$ MAYBE $= \infty$
- x^* OPTIMAL IF $\langle c, x^* \rangle \geq \langle c, x \rangle$ $\forall x$ FEASIBLE

EX UNOBTAINABLE FINITE VALUE

$$\begin{array}{ll} \text{MIN} & x \\ \text{S.T.} & z = 1 \\ & (x, y, z) \in \mathcal{Q} \end{array} \quad \xrightarrow{z=1 \text{ SUBST.}} \quad \begin{array}{ll} \text{MIN} & x \\ \text{S.T.} & x \geq 0 \\ & xy \geq 1 \end{array}$$

$\hookrightarrow$ VALUE $= 0$ BUT NOT OBTAINABLE, NO OPT SOL.

AVOID TROUBLES WHEN (P) NOT FEASIBLE BUT
TINY PERTURBATION MAKES (P) FEASIBLE
(NOT THE CASE FOR (LP))

DEF (SUBFEASIBILITY)

(P) IS SUBFEASIBLE IF $\exists (x_k)_{k \in \mathbb{N}} \subseteq K$
AND $\exists (x'_k)_{k \in \mathbb{N}} \subseteq L$ SUCH THAT
$$\lim_{k \to \infty} (A(x_k) + x'_k) = b$$

x_k, x'_k ARE SUBFEASIBLE SEQUENCES

VALUE OF SUBFEASIBLE SEQUENCE:
$$\langle c, (x_k)_k \rangle := \limsup_{k \to \infty} \langle c, x_k \rangle$$

SUBVALUE OF (P) $= \sup_{\text{FEASIBLE SEQ.}} \{ \langle c, (x_k)_k \rangle \}$

EX 

! VALUE OF (P) $\leq$ SUBVALUE OF (P),
(EVEN IF BOTH FINITE, STILL $<$ CAN BE)

DEF: $x \in K$ IS INTERIOR POINT IF $b - A(x) \in \mathrm{int}(L)$

THEOREM: (P) HAS INTERIOR POINT $\Rightarrow$
$\Rightarrow$ VALUE = SUBVALUE

DUAL PROGRAM

$$(D)\begin{cases} \min \langle l, y\rangle \\ \text{s.t.} \quad A^T(y) - c \in K^* \\ \qquad y \in L^* \end{cases}$$

$\therefore$ DUAL OF (D) IS (P)

SERIES OF THEOREMS;

THEOREM: WEAK DUALITY

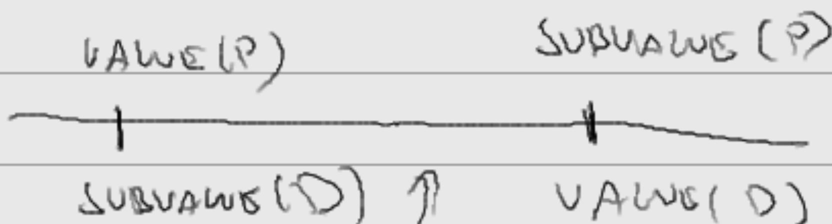
IF (D) FEASIBLE & (P) SUBFEASIBLE $\Rightarrow$
SUBVALUE(P) $\leq$ VALUE (D) $\leftarrow$ EASY

THEOREM: REGULAR DUALITY

(D) IS FEASIBLE & HAS FINITE VALUE $\beta \Rightarrow$
$\Rightarrow$ (P) IS SUBFEASIBLE & HAS FINITE SUBVALUE γ, MOREOVER, $\gamma = \beta$

$\hookrightarrow$ DONE FOR LP, FROM FARKAS LEMMA

BUT STILL



GAP := -C

THEOREM STRONG DUALITY

IF (P) FEASIBLE, FINITE VALUE γ, AND HAS INTERIOR POINT THEN (D) IS FEASIBLE AND HAS FINITE VALUE $\gamma = \beta$.

(INT POINT => VALUE = DUAL VALUE)

FOR $\overline{\underline{SDP}}$:

NOTE: IF $L = \{\emptyset\}$ DOES NOT APPLY: C NO INT.

THEOREM:

IF
$$(P)\begin{cases} \max & \langle c, x \rangle \\ \text{s.t.} & A(x) = b \\ & x \in K \end{cases}$$

SATISFY

IS FEASIBLE, HAS FINITE VALUE γ, $X \in \operatorname{int}(K)$

THEN
$$\begin{aligned} &\min \langle b, y \rangle \\ &\text{s.t. } A^T(y) - c \in K^* \end{aligned}$$

IS FEASIBLE AND HAS FINITE VALUE $\beta = \gamma$

LEMMA: $(PSD_n)^* = PSD_n$

EXERCISE