COLORING 3-COLORABLE GRAPHS

- LET $G=(V,E)$ BE A GRAPH

DEF:

- INDEPENDENT SET .. $S \subseteq V$ S.T. NO EDGE HAS BOTH VERTICES IN S
- $\alpha(G) = \max_{S \text{ INDEPENDENT}} |S|$
- COLORING: $c: V \to C$ ← COLORS S.T. $\forall uv \in E \cdot c(u) \neq c(v)$

$\boxed{\text{OR}}$ $V = S_1 \cup S_2 \cup .. \cup S_k$ — S_i IS INDEPENDENT SET

$|C|, k$ ... # OF COLORS

- G K-COLORABLE — $\exists$ COLORING S.T. $|C|=k$

PROBLEM:

INPUT K-COLORABLE G

OUTPUT COLORING OF G.

WE INTERESTED IN CASE $K = 3$

THM

- IF $P \neq NP$ THEN NO PTIME ALGORITHM CAN COLOR EVERY 3-COLORABLE GRAPH WITH AT MOST 4 COLORS
- IF UNIQUE GAME CONJECTURE THEN NO PTIME ALGORITHM CAN COLOR 4-COLORABLE GRAPH WITH 10000 COLORS

ANY FIXED CONSTANT

FOR 3-COLORABLE:

-> CAN WE DO AT LEAST SOMETHING?

- USING n COLORS - EASY
- USING $O(\sqrt{n})$ COLORS VIA GDERSON'S TRICK
  - NEIGHBORHOOD OF A VERTEX IS BIPARTITE => => 2-COLORABLE (EASY)
  - GIVEN Δ, USE ↑ ON MAX DEG. VERTEX WHILE MAX DEG $\geq \Delta$. AT MOST $O(n/\Delta)$ TIMES (& $3 \cdot O(n/\Delta)$ COLORS)
  - G OF MAX DEGREE Δ IS $\Delta+1$ COLORABLE
  - PICK $\Delta = \sqrt{n} \leadsto O(n/\sqrt{n}) + \sqrt{n} = O(\sqrt{n})$.

- COULD BE IMPROVED TO $n^{0.375}$ BY MORE TRICKS

KARGER, MOTWANI, SUDAN

- USING SDP $\tilde{O}(n^{0.25})$ 1998

$\tilde{O}(n^{0.211})$ ← 2006 BETTER ROUNDING

$f(n) = \tilde{O}(g(n))$ MEANS $f(n) \le g(n)(\log n)^{O(1)}$

FOR SDP:

DEF:

NOT JUST IN

FOR $K \in \mathbb{R}$ A STRICT VECTOR COLORING OF A GRAPH $G = (V, E)$ IS A MAPPING $\gamma : V \to S^{n-1}$ SUCH THAT

$$\gamma(u)^T \gamma(v) = -\frac{1}{K-1} \quad uv \in E$$

→ VECTORS FOR ADJACENT VERTICES HAVE LARGE ANGLE.

LEMMA IF G HAS K-COLORING, IT ALSO HAS STRICT VERTEX K-COLORING

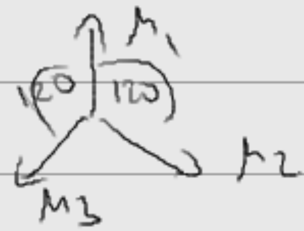
PROOF:

CONSTRUCT $\mu_1 \dots \mu_k$ S.T.

$$\mu_i^T \mu_j = -\frac{1}{k-1}$$

PICK μ_i AS VERTICES OF REGULAR SIMPLEX WITH ORIGIN AS CENTER

FIGURE:



FORMALLY $\mu_i = \dfrac{e_i - \frac{1}{k}\sum e_\ell}{\| e_i - \frac{1}{k}\sum e_\ell \|}$ □

DEF: (FOR SDP)

NON-STRICT VERTEX K-COLORING G. USE UNIT VECTORS S.T. $v_i^T v_j \le -\frac{1}{k-1}$ FOR $ij \in E$
(RELAX = TO $\le$)

- NOT CLEAR IF STRICT & NONSTRICT ARE DIFFERENT.

SMALLEST K COULD BE FOUND USING SDP

$\langle v_i, v_j \rangle \leq -X \in$ USING ROUNDING

THEOREM KARGER-MOTWANI-SUDAN

$\exists$ PTIME RANDOMIZED ALGORITHM S.T.

FOR G ON n VERTICES OF MAX DEGREE Δ

AND A VERTEX 3-COLORING OF G FIND

AN INDEP. SET OF EXPECTED SIZE $\tilde{\Omega}(\Delta^{-1/3} \cdot n)$

$\Rightarrow$ G COLORABLE BY $\frac{n}{\frac{n}{\Delta^{1/3}}} = \Delta^{1/3}$ COLORS

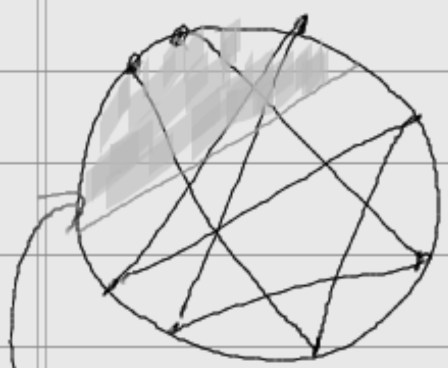
- USE WIGDERSON'S TRICK FOR

$\Delta = n^{3/4} \rightarrow \frac{n}{\Delta}$ STEPS $- n^{1/4}$ STEPS $\quad$ " $n^{1/4}$ COLORS

$\Rightarrow$ COLORING USING $\tilde{O}(n^{0.25})$

KMS ROUNDING: FOR GRAPH G

- MAX DEGREE Δ, $v_1 \ldots v_n \in \mathbb{R}^n$ IS VECTOR 3-COLORING ($\forall_{ij}$ EDGE : $\langle v_i, v_j \rangle \leq -\frac{1}{2}$)
- FIND LARGE INDEPENDENT SET IN G



IDEA: TAKE A CAP OF THE SPHERE.

PICK γ & t ← SUITABLE ON Δ

RANDOM

$\{x : \langle \gamma, x \rangle \ge t\} = I_0$

$I \subseteq I_0$ ISOLATED VERTICES $\Rightarrow$ I IS INDEP.

INTUITION: $i,j \in E$ $\Rightarrow$ UNLIKELY TO BE BOTH IN I.

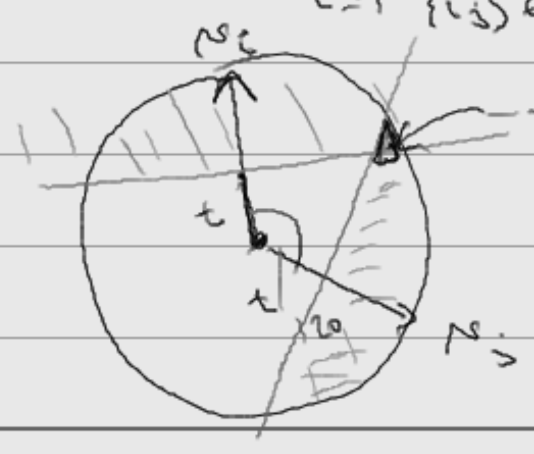
ANALYSIS:

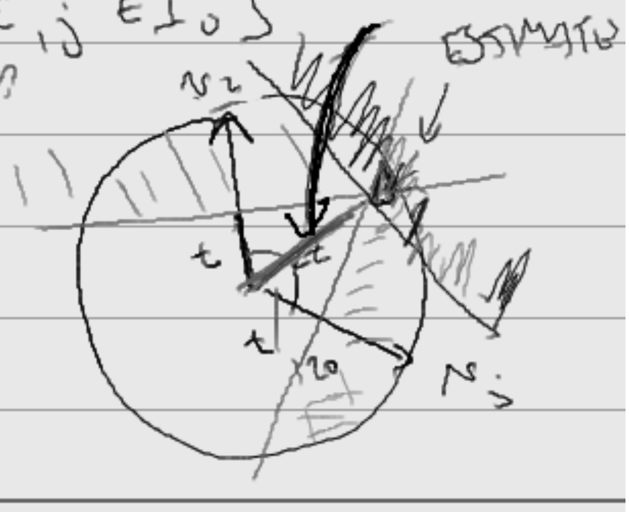
- $E[|I|] = E[|I_0|] - E[|I_0 \setminus I|]$
- $E[|I_0|] = \sum_{i=1}^{n} P[i \in I_0] = \sum P(\langle \gamma, v_i \rangle \ge t)$

$$E[|I_0|] = \sum_{i=1}^{n} P[i \in I_0 \ \& \ j \in I_0 \text{ FOR } ij \in E]$$

$$\le \sum_{i=1}^{n} \sum_{\{i,j\} \in E} P[i,j \in I_0]$$

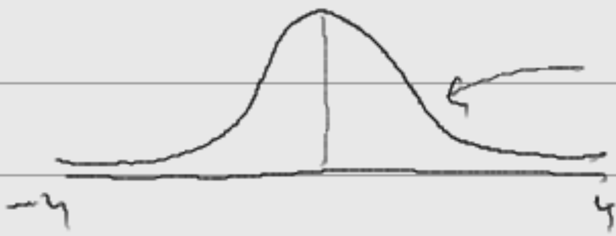
?





2t

PICKING g: USE NORMAL DISTRIBUTION IN EACH COORDINATE. $N(0,1)$



$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

BELL CURVE

IF u IS UNIT VECTOR, THEN

$\langle u, g \rangle \sim N(0,1)$

$N(t) := P[Z \geq t]$ $\to$ Z FROM $N(0,1)$

ESTIMATES:

$$\left(\frac{1}{t} - \frac{1}{t^3}\right) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq N(t) \leq \frac{1}{t} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

- $E[|I_0|] = \sum_{i=1}^{n} P[i \in I_0] = \sum \overbrace{P(\langle g, u_i \rangle \geq t)}^{\leq N(t)} = nN(t)$

$$E[|E_0|] \leq \sum_{i=1}^{n} \underbrace{\sum_{\{i,j\} \in E} \overbrace{P[i,j \in I_0]}^{\leq N(2t)}}_{\leq \Delta N(2t)}$$

$\Rightarrow E[|I|] \geq n\left(N(t) - \Delta N(2t)\right)$ USING

AND $t := \left(\frac{2}{3} \ln \Delta\right)^{1/2} \Rightarrow E[|I|] = \tilde{\Omega}(\Delta^{-1/3} \cdot n)$

CAN BETTER ROUNDING HELP? NO

CLAIM

$\exists$ CONSTANT $\delta > 0$ S.T. FOR INFINITELY MANY VALUES OF n, $\exists G$, $|V(G)| = n$, G IS VECTOR 3-COLORABLE, BUT NOT EVEN n^{δ}-COLORABLE.

$\Rightarrow$ JUST VECTOR 3-COLORING CANNOT GET "VERY GOOD" COLORING.