Math-484 Homework #2 (semidefinite and coercive)

I will finish the homework before 11am Sep 11. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am C14 (4 hours student).

1: (*What is positive/negative (semi)definite?*)
Try to Decide if the following matrices are positive or negative (semi)definite or indefinite using principal minors and explain why:
(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{pmatrix}$
(c) $\begin{pmatrix} -4 & 0 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -5 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2: (*I will recall what is a quadratic form.*)
Write the quadratic form $Q_A(\mathbf{x})$ associated to matrix

$$A = \begin{pmatrix} -4 & 0 & 4 \\ 0 & -3 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

3: (*I will recall what is coercive.*)
Decide which of these functions $\mathbb{R}^3 \to \mathbb{R}$ are coercive (of course, argue why):
(a) $f(x, y, z) = x^3 + y^3 + z^3 - xy$ (b) $f(x, y, z) = x^4 + y^4 + z^2 - 3xy - z$
(c) $f(x, y, z) = x^4 + y^4 + z^2 - xyz^2$ (d) $f(x, y, z) = x^4 + y^4 - 2xy^2$

4: (*Do I understand the assumptions of Theorem 1.3.3?*)
Show that the principal minors of the matrix

$$A = \begin{pmatrix} 1 & -8 \\ 1 & 1 \end{pmatrix}$$

are positive, but there are $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^2$ such that $\mathbf{x} \cdot A\mathbf{x} < 0$. Why does this not contradict Theorem 1.3.3 in the textbook?

5: (*Can I use all that stuff to find minimizers and maximizers?*)
Find (local, global) minimizers and maximizers of the following functions:
(a) $f(x_1, x_2) = e^{-(x_1^2+x_2^2)}$ (b) $f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$

6: (*I will check the definition of semidefinity more closely.* ***C14 only***)
Suppose that A is a $n \times n$-symmetric matrix for which $a_{ii}a_{jj} - a_{ij}^2 < 0$ for some $i \neq j$. Show

that A is indefinite.
Hint: See (1.3.4)(c) in the textbook.

7: (*A bit more coercive thinking.* ***C14 only***)
Find a continuos function $f(x, y)$ on $\mathbb{R}^2$ such that for each real number t, we have

$$\lim_{x \to +\infty} f(x, tx) = \lim_{y \to +\infty} f(ty, y) = +\infty$$

but such that $f(x, y)$ is not coercive.