Math-484 Homework #3 (convex sets and semidefinite matrices)

I will finish the homework before 11am Sep 18. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am C14 (4 hours student).

1: (*Local minimizer does not have to be global even if it is unique.*)
Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^3 + e^{3y} - 3xe^y.$$

Show that f has exactly one critical point and that this point is a local minimizer but not a global minimizer.

2: (*More exercises for deciding if matrix is (positive/negative,semi)definite*)
(a) Decide if the following matrix is (positive/negative,semi)definite:

$$A = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

(b) Decide for which $t \in \mathbb{R}$ is the following matrix is positive definite:

$$B = \begin{pmatrix} t & 1 & 0 \\ 1 & t & 1 \\ 0 & 1 & t \end{pmatrix}$$

3: (*Decomposition showing matrix is positive (semi)definite.*)
Suppose that A is a square matrix and suppose that there is another matrix B such that $A = B^T B$.
(a) Show that A is positive semidefinite.
(b) Show that if B has full column rank then A is positive definite.

4: (*Do I understand the definition of a convex set?*)
Are the follwing sets D in $\mathbb{R}^2$ convex?
(a) $\mathbf{x} = (100, 14) \in \mathbb{R}^2, \mathbf{y} = (15, 24) \in \mathbb{R}^2$.
$D = \{\mathbf{w} \in \mathbb{R}^2 : \mathbf{w} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{y},\text{ where } 0.3 < \lambda \leq 0.7\}$
(b) $D = B((1, 0), 1) \cup (0, 0)$ recall $B(\mathbf{x}, r) = \{\mathbf{w} : ||\mathbf{x} - \mathbf{w}|| < r\}$
(c) $D = B((1, 1), 1) \cup (0, 0)$
Note that we use ball as an open set in this exercise.

5: (*Space of semidefinite matrices is convex.*)
Let $\mathcal{PS}$ be the set of all positive semidefinite definite matrices in $\mathbb{R}^{n\times n}$. Show that $\mathcal{PS}$ is a convex set. Moreover, show that it is a closed set.
Upadte: For a matrix A use $\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}$.

6: (*Can I formally use convex combinations?*)
For $D \subseteq \mathbb{R}^n$ we define $co(D)$ to be the intersection of all convex sets containing D. Prove Theorem 2.1.4: Let $D \subseteq \mathbb{R}^n$. Then $co(D)$ coincides with the set C of all convex combinations of vectors from D.
Hint: 1) Show that C is a convex set containing D.
2) Show that if B is a convex set containing D then it also contains C.
3) Conclude that $co(D) = C$.

7: (*Piece of Cholesky factorization.* ***C14 only***)
If $M \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix, it can be decomposed as $M = B^T B$. Such decomposition can be obtained using Cholesky factorization (an algorithm running in $O(n^3)$). Prove the following observation that is a key ingredient of the factorization.
If the matrix

$$M = \begin{pmatrix} \alpha & \mathbf{q}^T \\ \mathbf{q} & N \end{pmatrix}$$

is positive semidefinite with $\alpha > 0$, then the matrix

$$N - \frac{1}{\alpha}\mathbf{q}\mathbf{q}^T$$

is also positive semidefinite. Note that $\alpha \in \mathbb{R}$, $\mathbf{q} \in \mathbb{R}^{n-1}$ and $N \in \mathbb{R}^{n-1\times n-1}$.