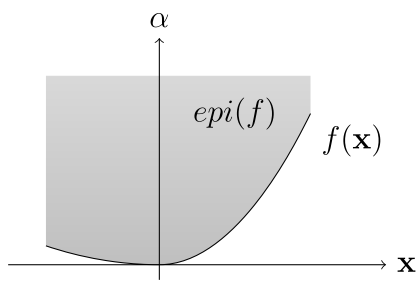
Math-484 Homework #4 (convex functions, $(A-G)$ inequality, and some repetition)

I will finish the homework before 11am Sep 25. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I am C14 (4 hours student).

1: (*I want to know what is* epigraph.)
Let $D \subset \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}$. The *epigraph* of f is a subset of $\mathbb{R}^{n+1}$ defined by

$$epi(f) = \{(\mathbf{x}, \alpha) : \mathbf{x} \in D, \alpha \in \mathbb{R}, f(\mathbf{x}) \leq \alpha\}.$$



Intuitively, *epigraph* are vectors above the graph of f including the graph of f.
a) Sketch the epigraphs of functions
$f(x) = e^x$ for $x \in \mathbb{R}$
$f(x_1, x_2) = x_1^2 + x_2^2$ for $(x_1, x_2) \in \mathbb{R}^2$
b) Show that $f(\mathbf{x})$ is convex if and only if epi(f) is convex.
c) Show that if $f(\mathbf{x})$ and $g(\mathbf{x})$ are convex functions defined on a convex set C then $h(\mathbf{x}) := \max\{f(\mathbf{x}), g(\mathbf{x})\}$ is also a convex function on C by showing that

$$\text{epi}(\max\{f(\mathbf{x}), g(\mathbf{x})\}) = \text{epi}(f(\mathbf{x})) \cap \text{epi}(g(\mathbf{x})).$$

2: (*I will learn more about composition of functions and convexity*)
Prove the following statement:
If $f(\mathbf{x}) : C \to \mathbb{R}$ is a concave function and $g(y)$ be an increasing concave function defined on the range of $f(\mathbf{x})$ then $g(f(\mathbf{x}))$ is a concave function.
Hint: See Theorem 2.3.10 (c) and its proof.

3: (*I recall the definition of a convex function.*)
Determine whether the functions are convex, concave, strictly convex or strictly concave on the specified sets:
(a) $f(x) = \ln x$ for $x \in (0, +\infty)$

(b) $f(x) = |x|$ for $x \in \mathbb{R}$
(c) $f(x_1, x_2) = 5x_1^2 + 2x_1x_2 + x_2^2 - x_1 + 2x_2 + 3$ for $(x_1, x_2) \in \mathbb{R}^2$
(d) $f(x_1, x_2) = (x_1 + 2x_2 + 1)^8 - \ln((x_1x_2)^2)$ for $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > x_2 > 1\}$
(e) $f(x_1, x_2) = c_1x_1 + c_2/x_1 + c_3x_2 + c_4/x_2$ for $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$, where c_1, c_2, c_3, and c_4 are positive constants
Hints: (b) epigraph may be usefull, (d) use previous question

4: (*Can I use* $(A - G)$ *inequality?*)
Solve using $(A - G)$ inequality the following problems: (Get the value of objective function and compute (x^*, y^*, z^*))
a) Minimize $x^2 + y + z$ subject to $xyz = 1$ and $x, y, z > 0$
b) Maximize xyz subject to $3x + 4y + 12z = 1$ and $x, y, z > 0$
c) Minimize $3x + 4y + 12z$ subject to $xyz = 1$ and $x, y, z > 0$

5: (*A little test repetition*)
Define $f(x, y, z)$ on $\mathbb{R}^3$ as $f(x, y, z) = e^x + e^y + e^z + 2e^{-x-y-z}$. Show that $Hf(x, y, z)$ is positive definite at all points of $\mathbb{R}^3$. Find strict global minimizer of f.
Hint: You should get $(\frac{\ln 2}{4}, \frac{\ln 2}{4}, \frac{\ln 2}{4})$ *as the minimizer.*

6: (*A little test repetition*)
Show that no matter what value of a is chosen, the function $f(x_1, x_2) = x_1^3 - 3ax_1x_2 + x_2^3$ has no global maximizers. Determine the nature of the critical points of this function for all values of a.

7: (*Semidefinite matrices theoretically.* ***C14 only***)
Show that the matrix

$$A(x) = \begin{pmatrix} x^4 & x^3 & x^2 \\ x^3 & x^2 & x \\ x^2 & x & 1 \end{pmatrix}$$

is positive semidefinite for all $x \in \mathbb{R}$.
Hint: See page 79, ex. 13 and 14.