Math-484 Homework #7 (linear programming)

I will finish the homework before 11am Oct 16. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I a 4 credits student.

1: (*Can I formulate and solve an easy linear program by geometry?*)
You have \$12,000 to invest, and three different funds from which to choose. The municipal bond fund has a 7% return, the local bank's CDs have an 8% return, and the high-risk account has an expected (hoped-for) 12% return. To minimize risk, you decide not to invest any more than \$2,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. Assuming the year-end yields are as expected, what are the optimal investment amounts?

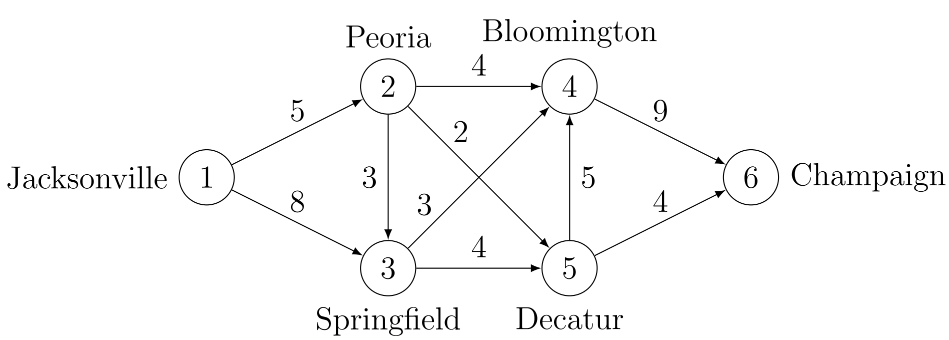
Formulate the question using linear programming using two variables and solve the program by examining the set of the feasible solutions (plot the set of feasible points in 2D). Invest all money you have.

2: (*Can I formulate and solve a linear program?*)
Convert the following program to a linear program (i.e. avoid using absolute values).

$$(P)\begin{cases}\text{Minimize} & |x|+|y|+|z| \\ \text{subject to} & x+y\le 0 \\ & 2x+z=3\end{cases}$$

and then solve it using `http://apmonitor.com`. Include the converted (LP) as well as the program for apmonitor (printout).

3: (*Can I formulate and solve a linear program?*)
A telephone company is would like to know how many simultaneous calls can be routed through their network from Jacksonville to Champaign. Their lines form the following network:



Each cable has a capacity of how many simultaneous calls it can handle (numbers on edges in the figure). Also calls can be directed only in the direction of the arrow. Any amount of calls can be directed through a city.
Formulate the problem as a linear program and solve using `http://apmonitor.com`. Include also printout of a program for APMonitor.
Hint: Use variables x_{ij} for the number of calls directed from city i to city j.

4: (*Can I formulate and solve a linear program?*)
A paper mill manufactures rolls of paper of a standard width 3 meters (=300 cm). But customers want to buy paper rolls of shorter width, and the mill has to cut such rolls from the 3 m rolls. One 3 m roll can be cut, for instance, into two rolls 93 cm wide, one roll of width 108 cm, and a rest of 6 cm (which goes to waste). Let us consider an order of
- 97 rolls of width 135 cm,
- 610 rolls of width 108 cm,
- 395 rolls of width 93 cm, and
- 211 rolls of width 42 cm.

What is the smallest number of 3 m rolls that have to be cut in order to satisfy this order, and how should they be cut?

Formulate the problem as a linear program. Write the program using language for `http://apmonitor.com` and solve it using APMonitor. The solution might not be satisfying - explain what is wrong with it and then write the program as integer linear program and give the correct solution.
Include also printout of the linear program as well as the linear integer program and printout of solution for the integer program.

5: (*What is a convex cone?*)
Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. A *convex cone* generated by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is

$$C = \{x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n : x_1, x_2, \ldots, x_n \geq 0\}.$$

Show that every convex cone is indeed convex. Prove, if it is closed or open. Draw a convex cone generated by $a_1 = (1,1), a_2 = (-1,1)$. Draw a convex cone generated by $a_1 = (1,1), a_2 = (-1,1), a_3 = (0,-1)$.

6: (*What is a Farkas lemma?*)
Farkas lemma:
Let $A \in \mathbb{R}^{m \times n}$, and let $b \in \mathbb{R}^m$ be a vector. Then exactly one of the following two possibilities occurs:
(F1) There exists a vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.
(F2) There exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T\mathbf{b} < 0$.

Prove Farkas lemma using convex cones and the separation theorem.
Hint: Interpret $A\mathbf{x}$ as a convex cone C from question 1. (F1) or (F2) corresponds to $b \in C$ and $b \notin C$. If $b \notin C$, then separation theorem gives a separating hyperplane. The hyperplane

will give y and the last thing to show is that the hyperplane indeed separates exactly as in $(F2)$. Also, see Farkas lemma notes on class webpage.