Math-484 Homework #8 (KKT and duality)

I will finish the homework before 11am Oct 30. If I spot a mathematical mistake I will let the lecturer know as soon as possible.
I will write clearly and neatly as the grader is not an expert in cryptography. I will sign each paper of my work and indicate if I a 4 credits student.

1: (*Can I use KKT?*)
Apply the Karush-Kuhn-Tucker Theorem to locate all solutions of the following convex programs:

$$(P_a)\begin{cases}\text{Minimize} & f(x_1,x_2)=e^{-(x_1+x_2)}\\ \text{subject to} & e^{x_1}+e^{x_2}\le 20\\ & x_1\ge 0\end{cases}\qquad (P_b)\begin{cases}\text{Minimize} & f(x_1,x_2)=x_1^2+x_2^2-4x_1-4x_2\\ \text{subject to} & x_1^2-x_2\le 0\\ & x_1+x_2\le 2\end{cases}$$

2: (*More KKT*)
Apply the Karush-Kuhn-Tucker Theorem to locate all solutions of the following convex program:

$$(P)\begin{cases}\text{Minimize} & -x_1+x_2\\ \text{subject to} & x_1^2+x_1-x_2-2\le 0\\ & 11x_1+5x_2-6\le 0\end{cases}$$

3: (*Can I solve geometric program?*)
Consider the following geometric program:

$$(GP)\begin{cases}\text{Minimize} & f(t_1,t_2)=t_1^{-1}t_2^{-1}\\ \text{subject to} & \frac{1}{2}t_1+\frac{1}{2}t_2\le 1\\ \text{where} & t_1>0,\ t_2>0\end{cases}$$

a) Convert (GP) to an equivalent convex program and solve the resulting program using KKT.
b) Solve the given (GP) by using methods of Chapter 5.3.

4: (*More geometric programming.*)
Solve the following geometric program:

$$(GP)\begin{cases}\text{Minimize} & x^{1/2}+y^{-2}z^{-1}\\ \text{subject to} & x^{-1}y^2+x^{-1}z^2\le 1\\ \text{where} & x>0,\ y>0,\ z>0\end{cases}$$

5: (*Can I solve simple things?*)
Let $f(x)$ be a differentiable function on $\mathbb{R}$. Suppose x_0 is fixed and there exists a number $\alpha \in \mathbb{R}$ such that

$$f(x) \geq f(x_0) + \alpha(x - x_0)$$

for all $x \in \mathbb{R}$. Show that $\alpha = f'(x_0)$.

6: (*Minimum norm solution and KKT* ***4 credits only***)
Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$ be a fixed vector. Suppose that convex program

$$(P) \begin{cases} \text{Minimize} & \|\mathbf{x}\|^2 \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b} \end{cases}$$

is super consistent and has a solution $\mathbf{x}^*$. Use Karush-Kuhn-Tucker Theorem to show that there is a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x}^* = A^T\mathbf{y}$.