MATH413 HW 9

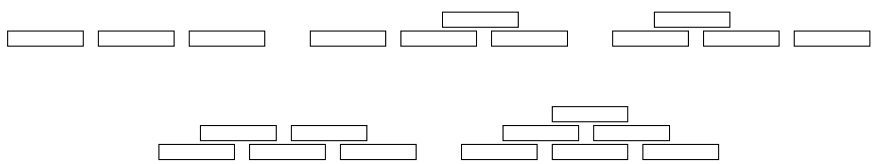
due **April 30** before class, **answer without justification will receive 0 points**. Staple all your papers.

1: Count permutations $a_1a_2\cdots a_n$ of $[n]$ for which there does not exist $i<j<k$ and $a_k<a_i<a_j$ (they are called 231-avoiding permutations).
Example of all counted permutations for $n=3$:

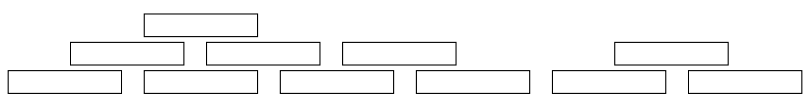
$$123 \quad 132 \quad 213 \quad 312 \quad 321$$

For example 536214 is not a 231-avoiding permutation. It is witnessed by $i=1, j=3, k=6$, which gives $(a_k=4)<(a_i=5)<(a_j=6)$.
(Hint: Where is n in the permutation?)

2: Let a_n be the number of configurations of pennies on a base row on n pennies, where pennies can be added so that each one not in the base rests on two in the row immediately below it. Give a formula for a_n.
Example: there are 5 configurations for $n=3$



An example of configuration for $n=6$ that is also counted:



(You may assume that a penny is always right in the middle of the two pennies below, as in the figures.)
(Hint: Paths or first place without any penny.)

3: Let $P(n)$ be a polynomial of degree 3 in n. Values of $P(n)$ for $n\in(0,1,2,3,4)$ are

$$-1,\ -1,\ 1,\ 11,\ 35.$$

Find $P(n)$.

4: Using the difference sequence method, find a closed formula for

$$\sum_{k=2}^{n} k^3-2k^2+1.$$

Verify your answer for $n=2$.
(Notice the limits for k in the sum.)

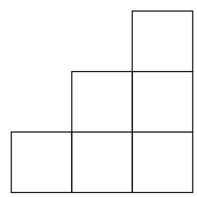
5: Let $[x]_n = x \cdot (x-1) \cdot (x-2) \cdot (x-3) \cdots (x-n+1)$ and $S(n,k)$ be the Stirling number of the second kind. Show that

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k.$$

(Hint: One possibility is to count mappings from $\{1,\ldots,n\}$ *to* $\{1,\ldots,x\}$*.)*

6: Prove that the Stirling numbers of the first kind satisfy the following formulas:
(a) $|s(n,1)| = (n-1)!$ for $n \geq 1$
(b) $|s(n,n-1)| = \binom{n}{2}$ for $n \geq 1$

7: Count the number of ways to place k rooks on $n \times n$ triangular board, i.e, square board that is missing places above the diagonal.
Example of board for $n = 3$.



(Hint: Find bijection to putting objects to boxes.)