Spring 2015, MATH-201

Final exam practice problems

This does not contain all/every problem that could/will be on final! Just a sample of problems you may expect.

1: Negate the following formula:

$$\forall x > 0, \exists z < 7, (z < x) \Rightarrow (x > 7)$$

2: Use truth table to check if the following formula is a tautology (recall that formula is a tautology if it is always true).

$$((A \Rightarrow B) \wedge \neg B) \Rightarrow (B \Rightarrow A)$$

3: Write as a logic formula, negate and write in English:
There exists a real number a for which $a + x = x$ for every real number x.

4: Use direct proof to show the following:
Suppose a is an integer. If $7|4a$, then $7|a$.

5: *4.17* Use direct proof to show that if two integers have opposite parity, then their product is even.

6: *5.11* Use contrapositive proof to show the following:
Suppose $x, y \in \mathbb{Z}$. If $x^2(y + 3)$ is even, then x is even or y is odd.

7: *6.9* Use proof by contradiction to show the following:
Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

8: *7.?* Prove that $5^{\frac{1}{2}}$ is irrational.

9: *7.11* Prove the following statement:
There is a set X for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

10: *7.20* Prove the following statement:
There exists an $n \in \mathbb{N}$ for which $11|(2n - 1)$.

11: *8.15* Prove the following statement in two ways. First by using Venn's diagrams and then without using Venn's diagrams (by (re)writing them as sets):
If A, B and C are sets, then $(A \cap B) - C = (A - C) \cap (B - C)$.

12: *9.17* Prove or disprove the following statement:
For all sets A and B, if $A - B = \emptyset$, then $B = \emptyset$.

13: *9.34* Prove or disprove the following statement:
If $X \subseteq A \cup B$,then $X \subseteq A$ or $X \subseteq B$.

14: *10.11* Prove by induction the following:
For any integer $n \geq 0$, it follows that $3|(n^3 + 5n + 6)$.

15: *10.25* Concerning the Fibonacci sequence, prove that $F_1 + F_2 + F_3 + F_4 + \ldots + F_n = F_{n+2} - 1$.

16: Use induction to show that

$$1 - \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2^n}.$$

17: *11.1.15* Prove or disprove: If a relation is symmetric and transitive, then it is also reflexive.

18: *11.2?* Let $A = \{a, b, c\}$. Decide if relation $R = \{(a, a), (a, b), (b, b), (b, c), (b, a), (c, b), (c, c)\}$ on A is an equivalence relation.

19: Solve the following linear system over $\mathbb{Z}_3$:

$$\begin{aligned} x + 2y + z &= 2 \\ x + 2y + 2z &= 1 \\ 2x + y + 2z &= 0 \end{aligned}$$

Answer has to be numbers in $\mathbb{Z}_3$. Something like $x = \frac{13}{4}$ is not an acceptable solution.

20: *12.1.9* Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

21: *12.2.14* Consider the function $\theta : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$. Is θ injective? Is it surjective? Bijective? Explain.

22: *12.6.11* Given $f : A \to B$ and subsets $Y, Z \subseteq B$,prove $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$.

23: *13.?* Consider the following sets of numbers: $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$. Order them by inclusion and say which have same cardinalities.

24: *13.2.9* Prove or disprove: The set $\{0, 1\} \times \mathbb{N}$ is countably infinite.

25: *13.4.3* Let $\mathcal{F}$ be the set of all functions $\mathbb{N} \to \{0, 1\}$. Show that $|\mathbb{R}| = |\mathcal{F}|$.

26: *1.?* Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$. Define $Z = \{x + y : x \in X, y \in Y\}$. Show that

$$\sup Z = \sup X + \sup Y.$$

27: *2.1.13* Let $\{x_n\}$ be a convergent monotone sequence. Suppose there exists a $k \in \mathbb{N}$ such that

$$\lim_{n\to\infty} x_n = x_k.$$

Show that $x_n = x_k$ for all $n \geq k$.

28: Use the definition of limit to verify that

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

29: Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

does not converge.

30: *2.5.8* Show that $\sum \frac{(-1)^n}{n}$ converges. Hint: consider sum of two subsequent entries.

31: *2.5.11* Prove the limit comparison test. That is, prove that if $a_n > 0$ and $b_n > 0$ for all n, and

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty,$$

then either $\sum a_n$ and $\sum b_n$ both converge of both diverge.

32: *Exercise 2.5.3*

33: Show that if $\{x_n\}_n^\infty$ and $\{y_n\}_n^\infty$ are sequences, then

$$\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \geq \limsup_{n\to\infty} (x_n + y_n).$$

Hint: by contradiction