Spring 2015, MATH-201

Functions

Function f from A to B (denoted by $f : A \to B$) is a relation $f \subseteq A \times B$, where for every $a \in A$ exists exactly one $b \in B$ such that $(a, b) \in f$ (or in different notation $f(a) = b$).

1: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. Decide if the following relations are functions from A to B:

- $\{(1, a), (1, d), (3, b)\}$
- $\{(1, d), (2, c)\}$
- $\{(1, b), (2, b), (3, b)\}$

A is called *domain*, B is *codomain*, *range* of f is $\{f(a) : a \in A\}$ (all possible values of f)
Functions f and h are *equal* if $f = h$ as sets.
A function $f : A \to B$ is

- *injective* (one-to-one) if $\forall x, y \in A, x \neq y \implies f(x) \neq f(y)$
- *surjective* (onto) if $\forall b \in B, \exists a \in A, f(a) = b$ (range is equal to codomain)
- *bijective* if f is both injective and surjective

Technique for showing that $f : A \to B$ is

- injective: Assume $x, y \in A$ and $x \neq y$. Conclude that $f(x) \neq f(y)$. (Direct approach)
- injective: Assume $x, y \in A$ and $f(x) = f(y)$. Conclude that $x = y$. (Contrapositive)
- surjective: Assume $b \in B$. Conclude there is $a \in A$ such that $f(a) = b$.

2: Decide if the following functions are injective,surjective,bijective:

- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^3 + 1$
- $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $g(m, n) = (m + n, m + 2n)$

Composition: Let $f : A \to B$ and $g : B \to C$ be functions (codomain of f is the domain of g). The *composition* of f and g is denote by $g \circ f$ and it is a function $g \circ f : A \to C$ defined as $g(f(x))$ for all $x \in A$.

3: Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, 2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

Note that:

- Composition of functions is associative. That is if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f))$. Since both sides of the equation evaluates as $h(g(f(x)))$.
- Suppose $f : A \to B$ and $g : B \to C$. If both f and g are injective, then $g \circ f$ is injective. If both f and g are surjective, then $g \circ f$ is surjective.

Inverse function: Let $R \subseteq A \times B$ be a relation. The *inverse relation* R^{-1} of R is relation on $B \times A$ defined as $\{(b, a) : (a, b) \in R\}$. (Just swapping order in the ordered pairs in R).

4: Let $f : A \to B$ be a function. Let f^{-1} be its inverse relation. Show that if f^{-1} is a function, then f is bijective. (Hint: show that if f is not injective or not surjective, then f^{-1} is not a function (contrapositive).)

5: Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as $f(x) = 8x^3 - 1$. Find functions f^{-1} and $f^{-1} \circ f$.

Let $f : A \to A$ be a function where $f(a) = a$ for all $a \in A$. Then f is called the *identity function* on A. Identity function is usually denoted by id, id_A i_A (the A changes based on the set).
Notice: If $f : A \to B$ and f^{-1} is a function, then $f^{-1} \circ f$ is id.

Image and Premiage: Let $f : A \to B$ be a function. Let $X \subseteq A$. The *image* of X is set $f(X) = \{f(x) : x \in X\}$. Let $Y \subseteq B$. The *preimage* of Y is set $f^{-1}(Y) = \{x \in A : f(x) = y\}$.

6: Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = 2x^2$. Find $f(\{3, 4, 5\})$ and $f^{-1}(\{2, 8\})$.

7: Suppose $f : A \to B$ is a function. Let $W, X \subseteq A$,and $Y, Z \subseteq B$. Show that:

$$f(W \cap X) \subseteq f(W) \cap f(X) \qquad f(W \cup X) = f(W) \cup f(X)$$

$$f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z) \qquad X \subseteq f^{-1}(f(X))$$