Spring 2015, MATH-201

Cardinality of Sets

Do sets A and B have the same size? (is $|\mathbb{Z}| > |\mathbb{N}|$?)

Two sets A and B have the same cardinality (same size), written $|A| = |B|$, if there exists a bijective function $f : A \to B$.

Examples:

- $|\mathbb{N}| = |\mathbb{Z}|$ by creating a a bijective function
- $|\mathbb{N}| \neq |\mathbb{R}|$ by Cantor's diagonalization (by contradiction)

1: Decide if the following sets have the same cardinality. Give a bijection or argument why the sets have different cardinality

- $\mathbb{R}$ and $(0, 1)$
- $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$
- $\mathbb{Z}$ and $(0, 1)$

Let A be an infinite set. If $|A| = |\mathbb{N}|$, then A is *countably infinite*. Otherwise A is *uncountable*. A set A is *countable* if $|A| = |B|$, where $B \subseteq \mathbb{N}$.

Example: $\mathbb{Z}$ is countable, $\mathbb{R}$ is uncountable.

Notation: $|A| = \aleph_0$ (aleph-null). Features: $\aleph_0 + 7 = \aleph_0$, $2\aleph_0 = \aleph_0$, $\aleph_0^2 = \aleph_0$.

Note: Set is countable if it elements can be ordered and labeled by $1, 2, 3, \ldots$.

2: Show that $|\mathbb{Q}| = \aleph_0$. (Find ordering of $\mathbb{Q}$.)

3: Let A and B be countable sets. Show that $A \times B$ is countable. (Find ordering of $A \times B$.)

Comparing cardinalities: Let A and B be sets. We use $|A| < |B|$ is there is an injective function $A \to B$ but no bijection between A and B. We use $|A| \leq |B|$ if $|A| < |B|$ or $|A| = |B|$.

Theorem: Let A be any set. Then $|A| < |\mathcal{P}(A)|$. Recall that $\mathcal{P}(A)$ is the power set of A, set of all subsets of A.

4: Show that $|A| < |\mathcal{P}(A)|$ for every set A. (Find an injective function from A to $\mathcal{P}(A)$. Show there is no bijection by contradiction, suppose for contradiction there is a bijection $f : A \to \mathcal{P}(A)$ and investigate what happens with set $B = \{x \in A : x \notin f(x)\} \in \mathcal{P}$.)

Corollary: $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$ Bigger and bigger sets.

The Cantor-Bernstein-Schröeder Theorem: (Tool for showing $|A| = |B|$.) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$.

5: Use Cantor-Bernstein-Schröeder Theorem to show $\mathbb{R} = \mathcal{P}(\mathbb{N})$. Hint: For example find injective functions $\mathcal{P}(\mathbb{N}) \to (0, 1)$ and $(0, 1) \to \mathcal{P}(\mathbb{N})$. Use that there are countably infinitely many digits after the decimal point in real numbers.)

Continuum hypothesis: There is no set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$.

6: Find bijections between:

- Z and $S = \{x \in \mathbb{R} : \cos x = 1\}$
- $(0, 1)$ and $(0, 1]$
- $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$

7: Let A be a countably infinite set and let $B \subseteq A$ be infinite. Show that B is countably infinite.

8: If $U \subseteq A$, and U is uncountable, then A is uncountable.