Spring 2015, MATH-201

Sequences and limits - definitions - Chapter 2.1

A *sequence* is a function $x : \mathbb{N} \to \mathbb{R}$. We use x_n for $x(n)$ and we write the sequence as

$$\{x_n\}_{n=1}^{\infty}$$

Example: $\{n\}_{n=1}^{\infty} = 1, 2, 3, 4, \ldots$; $\{7\}_{n=1}^{\infty} = 7, 7, 7, 7, \ldots$ (constant sequence)

A sequence $\{x_n\}_{n=1}^{\infty}$ is *bounded* if exists $B \in \mathbb{R}$ such that $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Example: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is (not?) bounded, $\left\{-n^2\right\}_{n=1}^{\infty}$ is (not?) bounded

Convergence: A sequence $\{x_n\}_{n=1}^{\infty}$ is *converging* to a number $x \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such $|x - x_n| < \varepsilon$ that for all $n \geq M$.

$$\forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n > M, |x - x_n| < \varepsilon$$

We say x is a *limit* of $\{x_n\}_{n=1}^{\infty}$ and we write

$$\lim_{n \to \infty} x_n = x.$$

A sequence that converges is called *convergent* and other are *divergent.*

Intuition: As n grows, x_n is getting closer to x.

1: Write a formula that is true iff $\{x_n\}_{n=1}^{\infty}$ is divergent. (i.e. negate formula for convergent sequence)

2: Decide if the following sequences are bounded, convergent, divergent

- $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$
- $\left\{\frac{2n^2-1}{n^2+1}\right\}_{n=1}^{\infty}$
- $\{(-1)^n\}_{n=1}^{\infty}$

Claim: Convergent sequence has a unique limit.

Proof by contradiction: Let $\{x_n\}$ have two limits $x \neq y$. Consider $0 < \varepsilon < |x - y|$.

3: Show that every convergent sequence is bounded.

4: Show from the definition that $\lim_{n\to\infty} \frac{2n^2-1}{n^2+1} = 2$. For example show $|2 - \frac{2n^2-1}{n^2+1}| < \varepsilon$ for all large n.

Monotone sequence: A sequence $\{x_n\}$ is *monotone increasing* if $x_n \leq x_{n+1}$ for all n. It is *monotone decreasing* is $x_n \geq x_{n+1}$ for all n.
Example $\{2 - \frac{1}{n}\}_{n=1}^{\infty}$ is monotone increasing.
If $\{x_n\}$ is monotone increasing, then $\lim_n x_n = \sup\{x_n : n \in \mathbb{N}\}$.

Tail of a sequence: For a sequence $\{x_n\}_{n=1}^{\infty}$ a K-*tail* is $\{x_n\}_{n=K}^{\infty}$, where $K \in \mathbb{N}$.
Theorem: $\{x_n\}_{n=1}^{\infty}$ converges iff $\{x_n\}_{n=K}^{\infty}$ converges and the limits are equal.

5: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Is it true that if there exists K, such that K-tail is bounded, then $\{x_n\}_{n=1}^{\infty}$ is bounded?

Subsequence: Let $\{x_n\}$ be a sequence. Let $\{n_i\}$ be a strictly increasing sequence of natural numbers. Then $\{x_{n_i}\}_{i=1}^{\infty}$ is a *subsequence* of $\{x_n\}$.

6: Let $\{x_n\} = \{n^2\}$ and $\{n_i\} = \{2n\}$. Fill the following table

$\{n\}$	1	2	3	4	5
$\{x_n\}$	1	4			
$\{n_i\}$	2				
$\{x_{n_i}\}$	4				

7: Show that every bounded sequence has a subsequence that is convergent.

8: Decide if $\{\sin(n)\}_{n=1}^{\infty}$ bounded, convergent, divergent.

9: Let $\{x_n\}_{n=1}^{\infty}$ be monotone. Show that it is convergent if and only if its bounded.

10: Show that $\{x_n\}_{n=1}^{\infty}$ converges iff $\{x_n\}_{n=K}^{\infty}$ converges and the limits are equal.