Spring 2015, MATH-201

Chapter 2.3 Limit superior, limit inferior, and Bolzano-Weierstrass

limit superior and limit inferior: Let $\{x_n\}$ be a bounded sequence. Let $a_n = \sup\{x_k : k \geq n\}$ and $b_n = \inf\{x_k : k \geq n\}$. The sequence $\{a_n\}$ is bounded monotone decreasing and $\{b_n\}$ is bounded monotone increasing (more on this point below). Define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n \text{ and } \liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n.$$

Example $\{(-1)^n\}$, $\{(-1)^n \frac{n+1}{n}\}$

1: Compute a_n and b_n if $x_n = (-1)^n \left(\frac{n+1}{n}\right)$. Hint: What is a_2 and a_4?

2: Show that $\{a_n\}$ and $\{b_n\}$ are monotone.

Let $\{x_n\}$ be a bounded sequence. Define a_n and b_n as above. then
(i) $\limsup x_n = \inf\{a_n : n \in \mathbb{N}\}$ and $\liminf x_n = \sup\{b_n : n \in \mathbb{N}\}$.
(ii) $\liminf x_n \leq \limsup x_n$.

Let $\{x_n\}$ be a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ such that $\lim x_{n_k} = \limsup x_n$. Similarly, there exists a (perhaps different) subsequence $\{x_{m_k}\}$ such that $\lim x_{m_k} = \liminf x_n$.
Example for $\{(-1)^n \frac{n+1}{n}\}$

3: Let $\{x_n\}$ be a bounded sequence. Show that if $\liminf x_n = \limsup x_n$ then $\{x_n\}$ converges.

4: A bounded sequence $\{x_n\}$ is convergent and converges to x if and only if every convergent subsequence $\{x_{n_k}\}$ converges to x.

Infinite limits If we allow $\liminf$ and $\limsup$ to take on the values ∞ and $-\infty$.

The definitions $\limsup x_n = \inf\{a_n : n \in \mathbb{N}\}$, and $\liminf x_n = \sup\{b_n : n \in \mathbb{N}\}$, where $a_n = \sup\{x_k : k \geq n\}$ and $b_n := \inf\{x_k : k \geq n\}$ still apply.
If $\lim x_n = \infty$ we say x_n diverges to infinity.

Bolzano-Weierstrass theorem: Let $\{x_n\}$ be a bounded sequence of real numbers. Then there exists a convergent subsequence $\{x_{n_i}\}$.
Proof Idea: Construct sequences $a_n \leq x_n \leq b_n$ and use squeeze theorem on $\lim a_n = \lim b_n$.

Note: the proof generalizes to sequences $\mathbb{N} \to S$, where S is a bounded compact sets.

2.4 Cauchy sequences
A sequence $\{x_n\}$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall n \geq M, \forall k \geq M, |x_n - x_k| < \varepsilon.$$

5: Show that $\{\frac{1}{n}\}$ is a Cauchy sequence.

6: Show that $\{x_n\}$ is convergent iff it is a Cauchy sequence.
(We can check if a sequence converges without knowing its limit!)