Fall 2015, MATH-304

Chapter 2.3 - Combinations (Subsets) of Sets - Binomial coefficient

Let S be a set of n element. An *r-combination* is a subset of S of size r.
How many r-subsets are there? How many ways to pick r elements out of n? ($= C(n,r)$)

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Binomial coefficient is $\binom{n}{r}$.
Note: Binomial identities can often be proved by rewriting using factorials and manipulation (tedious, but often works) or by explaining what is being counted (needs idea, shorter, more elegant).

1: Show that $\binom{n}{r} = \binom{n}{n-r}$.

2: If a 5-card hand is chosen at random (out of 52 cards), what is the probability of obtaining a flush (all cards are the same suit?) How about a full house? (Three cards of the same kind, and two of another kind, e.g., three queens and two "4"s)
(Recall that probability of an event is $\frac{\text{\# of desired outcomes}}{\text{\#all outcomes}}$).

3: *Pascal's formula* $C(n,k) = C(n-1,k) + C(n-1,k-1)$ That it

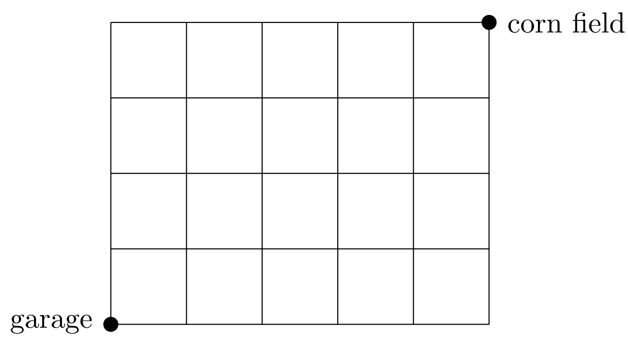
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

4: *Committee approach* Prove that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Think of finding a subset of a subset.

5: *Lattice paths* Count the number of ways drive a corn harvester from the garage (located at $0,0$) to a corn field, located at $(5,4)$.



Allowed movements are only right or up (you want to go the shortest path). (Try also counting in general to location (a,b), where $a,b \in \mathbb{Z}$)

6: Prove that

$$\sum_{0\le j\le b} \binom{a+j-1}{a-1} = \binom{a+b}{a}$$

Hint: Use lattice paths.

7: Prove that

$$\sum_{0\le j\le n} \binom{n}{j}^2 = \binom{2n}{n}$$

8: Prove that $\binom{a+b}{a}$ equals the number of partitions (a sequence of integers $P_1 \geq P_2 \geq \ldots \geq P_b$) satisfying $a \geq P_1 \geq P_2 \geq \ldots \geq P_b \geq 0$.

9: *Binary string* Prove that

$$\sum_{0 \leq k \leq n} \binom{n}{k} = 2^n$$

10: Prove that

$$\sum_{0 \leq k \leq n} \binom{k}{r} = \binom{n+1}{r+1}$$

11: Prove

$$\sum_{0 \leq k \leq r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$