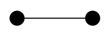
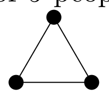
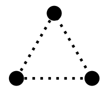
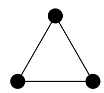
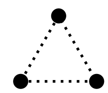
Fall 2015, MATH-304

# Chapter 3.3 Ramsey Theory - complete chaos is not possible

Suppose people at a party. Two know each other  or don't know each other 

**1:** Find a diagram of a party of 5 people such that no 3 people all know each other or do not know each other. That is, we don't see  or 

**2:** Is it possible to find a diagram on 6 people without  and  ?
(Hint: Pick one person and investigate who he knows. . . )

**Graph** is $G$ is a pair of vertices $V$ and edges $E$. That is $G = (V, E)$. Edges $E$ are pairs of vertices.
Example: People are vertices and if they know each other, add edge.

Complete graph $K_n$ is a graph on $n$ vertices with all possible edges.

**3:** Draw $K_n$ for $n \in \{1, 2, 3, 4, 5\}$.

Notation: $K_6 \to K_3K_3$ reads as ($K_6$ arrows $K_3K_3$) and means in every coloring of edges of $K_6$ by two colors, there exists either $K_3$ in the first color or $K_3$ in the second color.

Notice that edges and non-edges can be treated as 2 colors.

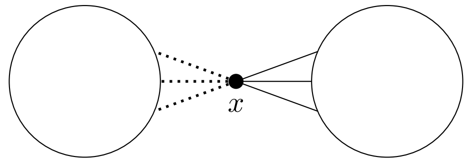
**Theorem (Ramsey)** $\forall m, n, \exists p$ such that $K_p \to K_mK_n$.

In other words, every 2-coloring of a huge graph $K_p$ contains a monochromatic $K_m$ or $K_n$.

Denote smallest $p$ by $r(m, n)$.

**4:** Determine $r(2, n)$.

**5:** Show that $r(m,n) \le r(m-1,n) + r(m,n-1)$. (Hint: Consider $p = r(m-1,n) + r(m,n-1)$ points. Pick any point $x$ and study set of blue or red neighbors.)




Ramsey's theorem can be extended to more than 2 colors. For $c$ colors, we have $K_p \to K_{n_1} K_{n_2} \cdots K_{n_c}$.

**6:** Show Ramsey's theorem for 3 colors. That is, prove that $r(m,n,o)$ is finite (minimum $p$ such that $K_p \to K_m K_n K_o$).

Ramsey's theorem can be extended to coloring more than pairs of vertices. For $c$ colors, we have $K_p^t \to K_{n_1}^t K_{n_2}^t \cdots K_{n_c}^t$, which means that if we color all $t$ subsets of vertices by $c$ colors, there exists $i$ such that there are $n_i$ vertices where all $t$-subsets have color $i$.

**Probabilistic lower bound by Erdős** $r(k,k) \ge \lfloor 2^{k/2} \rfloor$ for all $k \ge 3$.

Consider a random coloring of edges of $K_n$ by red and blue.

**7:** What is the number of edges of $K_n$?

**8:** What is the probability that a fixed set of $k$ vertices is red? (all edges are red)

**9:** What is the probability that a fixed set of $k$ vertices is monochromatic? (all edges red or blue)

**10:** What is the possible number of $k$-subsets?

**11:** What is the expected number of monochromatic subsets of size $k$? Recall expected value of $X$ is $EX = \sum_X p(X)X$.

**12:** Try to use $n = \lfloor 2^{k/2} \rfloor$ and give an upper bound on the expected value.

**13:** What happens if the upper bound is $< 1$?