Fall 2015, MATH-304

Chapter 5.1 Pascal's triangle

Recall Pascal's formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and $\binom{n}{0} = \binom{n}{n} = 1$.

1: Create a Pascal's triangle for n up to 8.

$n \diagdown k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1		1						
3	1			1					
4	1				1				
5	1					1			
6	1						1		
7	1							1	
8	1								1

2: Suppose you start at $(0,0)$ in Pascal's triangle. How many ways can you get to (n,k) if you can walk only in directions $\downarrow$ and $\searrow$?

Chapter 5.2 Binomial Theorem

Theorem 5.2.1 For every integer $n > 0$ and all x and y

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}y^n$$
$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i}y^i$$

3: Prove Binomial Theorem. (Hints: Investigate how the multiplication expands. Or think about an alphabet of $x+y$ letters and making n-letter words. Or induction.)

4: Expand $(x+1)^n$.

5: What happens with Binomial theorem if $x = 1$ and $y = 1$? Give a combinatorial interpretation of the resulting identity.

6: What happens with Binomial theorem if $x = 1$ and $y = -1$? Give a combinatorial interpretation of the resulting identity.

7: Prove that the sequence of numbers in each row of Pascal's triangle is a power of 11. I.e., "$1, 2, 1$" $\rightarrow$ $121 = 11^2$. For this you need to *carry over* numbers bigger than 9 to the left. So for example "1,5,10,10,5,1" gives "161051"$=11^5$.
Hint: How to build the result from numbers/digits in the triangle? By sum and multiplying by $10, 100, \ldots$*..?*

8: What is the coefficient of x^5y^{13} in the expansion of $(3x - 2y)^{18}$? What is the coefficient of x^8y^9?

9: Show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.
(Try to find also a combinatorial argument)

10: Show that

$$1\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}=n2^{n-1}$$

(Is it possible to do it from binomial theorem? Hint: derivative.)

11: Evaluate

$$\binom{n}{0}-2\binom{n}{1}+3\binom{n}{2}+\ldots+(-1)^n(n+1)\binom{n}{n}.$$

12: By integrating the binomial expansion, prove that, for any integer n,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$$