Fall 2015, MATH-304

Chapter 5.3 Unimodality of Binomial Coefficients

Let S be a set. Let $\mathcal{C} \subseteq P(S) = 2^S$ ($\mathcal{C}$ is a set of subsets).

$\mathcal{C}$ is a **chain** if $\forall A, B \in \mathcal{C}$ $A \subseteq B$ or $B \subseteq A$

$\mathcal{C}$ is an **antichain** if $\forall A, B \in \mathcal{C}$ $A \not\subseteq B$ and $B \not\subseteq A$

Example: Let $S = \{a, b, c, d, e\}$. $\mathcal{C} = \{\{a, b, c\}, \{b, c\}, \{c\}\}$ is a chain and
$\mathcal{C} = \{\{a, b\}, \{b, c\}, \{a, d, c\}\}$ is an aintichain.

We use Hasse diagram to draw the set of all subsets in inclusion relations.

(The diagram is used for Partially Ordered Sets (**poset**s). We use $A \leq B$ is $A \subseteq B$).

1: Let $|S| = n$. What is the size of the longest chain in $P(S)$?

2: How many longest chains are there in $P(S)$?

3: Let $\mathcal{C}$ be a chain and $\mathcal{A}$ be an antichain. What is the maximum size of $\mathcal{C} \cap \mathcal{A}$?
(Hint: What are possible sizes of intersection?)

4: Let $|S| = n$. Let $\mathcal{Y}$ be the set of all subsets that have size k. Is $\mathcal{Y}$ a chain or antichain?

5: Let $|S| = n$. What is the largest antichain in $P(S)$ that contains only sets of the same size?
(What is the largest binomial coefficient $\binom{n}{k}$ over all k?)

Sperner's theorem: Let $|S| = n$. Then the size of maximum antichain in $P(S)$ is at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof: Let $\mathcal{A}$ be the maximal antichain. Count the size of

$$X = \{(A, \mathcal{C}) : A \in \mathcal{A}, \mathcal{C} \text{ is a maximum chain}, A \in \mathcal{C}\}.$$

Note that $A \subseteq S$ and we are counting intersections of $\mathcal{A}$ with chains.

6: If $\mathcal{C}$ is a fixed maximum chain, how many pairs $(A, \mathcal{C})$ in X contain this chain? Does it give an upper bound on $|X|$?

7: Let $A \in \mathcal{A}$ be fixed. Suppose $|A| = k$. How many pairs $(A, \mathcal{C})$ in X contain A? (That is, how many maximum chains contain A?)

Let $a_k = |\{A \in \mathcal{A} : |A| = k\}|$. Notice that $|\mathcal{A}| = \sum_{k=0}^{n} a_k$.
The double counting of $|X|$ gives

$$\sum_{k=0}^{n} a_k k!(n-k)! = |X| \leq n!$$

8: Finish the proof of the Sperner's theorem by showing that $|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

9: Let $X = \{1, 2, \ldots, 9\}$. Let $(X, |)$ be a partial ordered set where $a \leq b$ if $a|b$ (means a divides b). Draw Hasse diagram for X and find a maximum chain and antichain.

10: Let $(X, \leq)$ be a poset. Suppose the size of the maximum chain is k. Show that $(X, \leq)$ can be partitioned into k antichains (parititon is disjoint union).