Fall 2015, MATH-304

Chapter 5.4 Multinomial Theorem

Binomial coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Multinomial coefficient: $\binom{n}{n_1\ n_2\ \cdots\ n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$, where $\sum_{i=1}^{k} n_i = n$.

1: Prove the following identity, which is a generalization of Pascal's formula for multinomial coefficients.

$$\binom{n}{n_1\ n_2\ \cdots\ n_k} = \binom{n-1}{n_1-1\ n_2\ \cdots\ n_k} + \binom{n-1}{n_1\ n_2-1\ \cdots\ n_k} + \cdots + \binom{n-1}{n_1\ n_2\ \cdots\ n_k-1}$$

Hint: Combinatorial verification might be more elegant.

Theorem 5.4.1. *(Multinomial theorem)* Let $n \in \mathbb{N}$. For all $x_1, \ldots, x_k$

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{n_1\ n_2\ \cdots\ n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $\sum$ is over all non-negative integral solutions of $n_1 + n_2 + \cdots + n_k = n$.

2: Prove the theorem by generalizing some of the proofs for binomial theorem.

3: What is the coefficient of $x_1^2 x_2 x_3^2$ in the expansion of $(2x_1 - 4x_2 - 3x_3)^5$?

Chapter 5.5 Newton's Binomial Theorem

Theorem 5.5.1 Let $\alpha \in \mathbb{R}$. Then for all $0 \le |x| < |y|$:

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k},$$

where $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$.

4: What happens when $\alpha \in \mathbb{N}$? What happens when $k > \alpha$?

5: Suppose $\alpha = -n$, where $n \in \mathbb{N}$. Evaluate $\binom{\alpha}{k} = \binom{-n}{k} = \cdots$ using normal binomial coefficient.

Restatement:

$$(x+y)^\alpha = y^\alpha \left(\frac{x}{y} + 1\right)^\alpha = y^\alpha (z+1)^\alpha$$

where $-1 < z < 1$. Assume $|z| < 1$:

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

6: Use Newton's Binomial Theorem to expand $\frac{1}{(1+z)^n}$ and evaluate for $n = 1$. (assume $|z| < 1$)

7: Use Newton's Binomial Theorem to expand $\frac{1}{(1-z)^n}$ and evaluate for $n = 1$. (assume $|z| < 1$)

8: Evaluate $\binom{\alpha}{k}$ for $\alpha = \frac{1}{2}$. Recall $\binom{\alpha}{0} = 1$ and try for $k > 0$.

9: Approximate $\sqrt{6}$ by using Newton's Binomial Theorem and

$$\sqrt{6} = \sqrt{4+2} = \sqrt{4(1+0.5)} = 2\sqrt{1+0.5} = \cdots$$