Fall 2015, MATH-304

Chapters 6.3 Derangements

Suppose a big crowd of people throw hats in the air. Everyone catches one hat at random. What is the probability that nobody has his/her own hat?

Formal version: Let S_n be permutations on $\{1, 2, \ldots, n\}$. Pick $\pi \in S_n$ uniformly at random. What is

$$P[\pi(i) \neq i, \forall i] =?$$

Permutation π, where $\forall i, \pi(i) \neq i$ is called a permutation **without fixed point**.
Let D_n be the number of permutations in S_n without fixed points.

1: Compute D_n using principle of inclusion and exclusion.

2: Compute $\lim_{n\to\infty} D_n/n!$. How fast does it converge?

3: Show that $D_n = (n-1)(D_{n-1} + D_{n-2})$. Hint: Think about $\pi(1)$ where $\pi \in S_n$ is without a fixed point.

4: Simplify the previous recurrence and prove that $D_n = nD_{n-1} + (-1)^n$. Hint slightly rewrite the previous recurrence and expand it.

5: Use the recurrence to compute D_5.

6: *A party with 7 gentlemen. How many ways to mix theirs hats such that nobody has his own?*

7: *A party with 7 gentlemen. How many ways to mix theirs hats such that at least one has his own?*

8: *A party with 7 gentlemen. How many ways to mix theirs hats such that at least two has their own?*

9: *Denote by $D_{n,k}$ the number of permutations in S_n with exactly k fixed points. Notice that $D_n = D_{n,0}$. Is it possible to express $D_{n,k}$ using D_m for suitable m?*

10: *(Bonus)* There are n canisters of gas distributed around a circular track which when all the gas is combined is exactly the amount needed for one car to make one lap of the track [the canisters are not all equally sized nor equally spaced]. Show that there is a location for a car to start with an empty tank (i.e., next to one of the canisters of gas) so that the car can make a full lap by collecting gas as it drives around the track.