Fall 2015, MATH-304

Chapters 6.4 Permutations with forbidden positions

Recall derangements: $\pi \in S_n$ such that $\pi(i) \neq i$.

Suppose that every i has a set of forbidden images X_i. That is, for all i we have $\pi(i) \notin X_i$. Use notation

$$P(X_1, X_2, \ldots, X_n) = \{\pi \in S_n : \forall i, \pi(i) \notin X_i\}$$
$$p(X_1, X_2, \ldots, X_n) = |P(X_1, X_2, \ldots, X_n)|.$$

1: Let $n = 3$ an $X_1 = \{2\}, X_2 = \{1,3\}, X_3 = \emptyset$. Write elements of $P(X_1, X_2, X_3)$ and compute $p(X_1, X_2, X_3)$.

2: Write derangements using the $P(X_1, X_2, \ldots, X_n)$ and $p(X_1, X_2, \ldots, X_n)$ notation.

The problem of permutations with forbidden positions can be formulated using placing non-attacking rooks on a board with forbidden positions.

3: Let $n = 4$ and $X_1 = \{1,2\}, X_2 = \{2,3\}, X_3 = \{3,4\}, X_4 = \{1,4\}$. Create an instance of placing non-attacking rooks on $n \times n$ board with forbidden positions and find a bijection between placing rooks and $P(X_1, X_2, X_3, X_4)$. This shows $p(X_1, X_2, X_3, X_4)$ is the number of placings rooks. Compute $p(X_1, X_2, X_3, X_4)$ (using principle of inclusion and exclusion, use A_i for rook i is placed in X_i, that is the bad positions).

Suppose we have an $n \times n$ board with forbidden positions and corresponding $X_1, \ldots, X_n$.

4: Let $A_i = \{\pi \in S_n : \pi(i) \in X_i\}$ be the bad permutations for i. Use principle of inclusion and exclusion to find a formula for $p(X_1, \ldots, X_n)$.

5: Is it possible to simplify $\sum_i |A_i|$ using X_is?

6: Write $|A_i \cap A_j|$ using X_i and X_j.

Use notation

$$\sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k \cdot (n-k)!$$

Theorem 6.4.1 Number of ways to place non-attacking rooks on board $n \times n$ with forbidden squares is

$$n! - r_1(n-1)! + r_2(n-2)! - r_3(n-3)! \cdots (-1)^n r_n.$$

7: Determine how many permutations on S_6 are where the forbidden images are $X_1 = \{1\}$, $X_2 = \{1, 2\}$, $X_3 = \{3, 4\}$, $X_4 = \{3, 4\}$, $X_5 = \emptyset$, $X_6 = \emptyset$.

Note: The method works well if the number of forbidden positions is small.

Chapters 6.5 Another Forbidden Position Problem

Problem: n boys take a walk in a line

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ \ldots\ n$$

where 1 precedes 2 who precedes 3 etc.
How many ways are there to rearrange the boys so that no one precedes the person he preceded before?
e.g., if $n = 3$ then 2 1 3 is OK but not 2 3 1 (2 is right in front of 3 as it was before)
Restated: count permutations $\Pi \in S_n$ that avoid the pairs

$$12, 23, ..., n-1n,$$

denote the number by Q_n.

8: Compute Q_1, Q_2 and Q_3. Brave may try Q_4.

9: Show a general formula

$$Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!.$$

Hint: Use principle of inclusion and exclusion, let A_j be permutations where $j(j+1)$ occurs, then use principle of inclusion and exclusion.

10: Show that $|A_{i_1}| = (n-1)!$.

11: Show that $|A_{i_1} \cap A_{i_2}| = (n-2)!$.

12: Show that $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$ for all k.

13: Use principle of inclusion and exclusion to compute Q_n.

14: *Problem des ménages* A host wants to seat n couples in a table, seating the men first. However, the host does not want to put wives on either side of their husband. How many ways are there to do this?

Hint: Use rooks placements. The resulting formula is from the principle of inclusion and exclusion but it is possible to compute the coefficients r_i.

15: *Bonus* Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Next time: Chapter 7.1