Fall 2015, MATH-304

Chapters 7.2 Generating Functions

Study sequences of numbers $h_0, h_1, h_2, \ldots$.
Say they count # of solutions to some combinatorial problem.
Idea: Use h_i as coefficients of a polynomial

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_t x^t + \cdots$$

Use notation

$$h_i = [x^i]g(x).$$

Trick: Sometimes it is easier to solve all h_i's at the same time than separately.

1: Find generating function for sequence

$$1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$$

2: Find generating function for sequence

$$1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \cdots$$

3: *Find generating function for sequence*

$$1\ 3\ 3\ 1\ 0\ 0\ 0\ 0 \cdots$$

Closed form of generating function is handy when multiplying or adding generating functions.

4: *Find coefficient of x^{2015} of $g(x)$, that is $[x^{2015}]g(x)$, where*
(a) $g(x) = (1-2x)^{5000}$
(b) $g(x) = \frac{1}{1+3x}$
(c) $g(x) = \frac{1}{(1+5x)^2}$

5: *Let $k \in \mathbb{N}$ be fixed. Let h_t be the number of integer solutions of*

$$e_1 + e_2 + e_3 + \cdots + e_k = t,$$

where $e_1 \geq 0, e_2 \geq 0, \ldots, e_k \geq 0$. Find a closed form for the generating function.

6: *Find a (more) closed form for the following generating function and try to find interpretation as solutions*

$$(1 + x + x^2 + x^3 + x^4 + x^5) \cdot (x + x^2) \cdot (1 + x + x^2 + x^3 + x^4)$$

7: *Write down the generating series for counting the number of possibilities to pay t cents using* 1, 5 *and* 25 *cent coins.*

8: *Count the number of ways to make a pack of n fruits if*

- *# of apples is even*
- *# of bananas is a multiple of 5*
- *# at most 4 oranges*
- *# 0 or 1 pear*

Write as generating function $g(x)$ and read $[x^n]g(x)$.

Notice the generating function works something like

(or or or) and (or or or) and (or or or)

9: *20 students, how many ways to pick 7 who get A? Solve this using generating functions as well as without it. Build the generating function be deciding for every student individually if the student is getting A or not.*

10: *20 students, how many ways to distribute 50 identical candies to the students? Use generating functions.*

11: *Compute generating function counting number of ways of getting sum h_i on two dices.*

12: *Determine the generating series for partitions. Partitions is the number of ways to write n as a sum on at most n non-negative integers that decrease in size. That is,*

$$n = x_1 + x_2 + x_3 + \cdots + x_n,$$

where $x_1 \geq x_2 \geq x_3 \geq \cdots \geq 0$. Let h_n be the number of partitions on n, write the generating function for sequence h_n.
Hint: In partition is important how many times is each number used. Solution is a big product.

13: Give generating function for the following sequence

$$1, 2, 3, 4, 5, 6, \ldots$$

Next time: Few more ordinary generating functions and then exponential generating functions.