Fall 2015, MATH-304

Chapter 7.3 Exponential Generating Functions

Recall ordinary generating functions:
Sequence 1 1 1 1 1 1 1 ... corresponds to $1 + x + x^2 + \ldots = \frac{1}{1-x} = g(x)$.

1: Find ordinary generating function for

$$1\ 1\ 2!\ 3!\ 4!\ 5!\ 6!\ \cdots$$

Definition: Exponential generating function is defined as

$$g^{(e)}(x) = \sum_{i=0}^{\infty} h_i \frac{x^i}{i!},$$

where $0! = 1$.

2: Find exponential generating function for

$$0!\ 1!\ 2!\ 3!\ 4!\ 5!\ 6!\ \cdots$$

3: Find exponential generating function for

$$1\ 1\ 1\ 1\ 1\ 1\ 1\ \cdots$$

4: Let $a \in \mathbb{N}$ what are coefficients at $e^{ax} = g^{(e)}(x)$? Find combinatorial explanation.

Note: Ordinary generating series are good for counting unlabeled (unordered) objects.
Exponential generating series are good for counting labeled (ordered) objects

5: What is the number of permutations of the word MISSISSIPPI?

Theorem 7.3.1 Let S be a multiset $= \{n_1 \times a_1; n_2 \times a_2; \cdots ; n_k \times a_k\}$, where $n_i \geq 1$. Let h_n be the number of n-permutations of S. Then

$$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) \cdots f_{n_k}(x),$$

where

$$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n_i}}{n_i!}$$

Holds also for $n_i = \infty$.

6: Prove theorem 7.3.1.

7: Let h_n be the number of n digit numbers with digits 1,2,3. (without restrictions, 3^n numbers). Suppose in addition, that

- # of 1s is even
- # of 2s is at least 3
- # of 3s is at most 4

Find generating function $g^{(e)}(x)$. [not a particularly nice]

8: Find closed form of exponential generating functions for

- 1 4 4^2 4^3 4^4 4^5 ...
- 1 -1 1 -1 1 -1 ...
- 1 0 1 0 1 0 ...

9: Let $n \in \mathbb{Z}$ be fixed. Define

$$h_k = \begin{cases} \frac{n!}{(n-k)!} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Find exponential generating function for h_k.

10: Count the number h_n of colorings of squares of a board $1 \times n$ by colors red, green, and blue such that the number of red squares is even. Solve using exponential generating functions.

11: Count the number h_n of colorings of squares of a board $1 \times n$ by colors red, green, and blue such that the number of red squares is even. Solve without using generating functions (use induction like (what color is the first square), find recurrence formula for h_n, use complement)

12: Determine the number of n-digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.