Chapter 7.4 Solving Linear Homogeneous Recurrence Relations

Definition Linear recurrence relation of order k is a sequence h_n defined as

$$h_n = a_1 h_{b-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n$$

where $a_k \neq 0$, $a_i \in \mathbb{R}$ and $h_0, h_1, \ldots, h_{k-1}$ are given. If $b_n = 0$, recurrence is called **homogeneous**.

1: Let Fibonacci numbers be defined as $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, and $F_1 = 1$.
Is it a linear recurrence relation? Is it homogeneous?

2: Let Derangements be defined as $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$, $D_1 = 0$, and $D_2 = 1$.
Is it a linear recurrence relation? Is it homogeneous?

Recall, Fibonacci numbers satisfy $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ and we obtained this by trying to solve $F_n = q^n$ for some constant q.

Goal: Solve *many* recurrence relations in form q^n.

Theorem 7.4.1 Let $q \neq 0$, Then $h_n = q^n$ is solution to

$$h_n = a_1 h_{b-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}$$

iff

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0 \text{ (this is called } \textit{characteristic equation}\text{)}$$

has q as a root, called *characteristic root*. If the characteristic equation has distinct roots $q_1, \ldots, q_k$, then

$$h_n = c_1 q_1^n + c_1 q_2^n + \cdots + c_k q_k^n,$$

where $c_i \in \mathbb{C}$. Constants c_i help with fitting the initial values of the sequence.

3: Prove Theorem 7.4.1

4: Solve the following recurrence relation:

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$$

with $h_0 = 1$, $h_1 = 2$, $h_2 = 0$.

5: Solve the following recurrence

$$h_n = h_{n-1} + 2h_{n-2}$$

with $h_0 = 2$ and $h_1 = 7$.

Solution using generating functions Idea: Find generating function $g(x)$ for h_n and then read $[x^n]g(x)$.

Recall

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} \binom{-n}{k}(-rx)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k$$

6: Solve the following recurrence using generating functions

$$h_n = 5h_{n-1} - 6h_{n-2}$$

$h_0 = 1$ and $h_1 = -2$.

7: Solve the following recurrence using generating functions

$$h_n = h_{n-1} + 2h_{n-2}$$

with $h_0 = 2$ and $h_1 = 7$.