MATH-566 HW 7

Due **Oct 14** before class. Just bring it before the class and it will be collected there.

1: (*Alternative attempt to define minimum spanning tree as LP*)
Let $G = (V, E)$ and $|V| = n$.

Recall that the spanning tree polytope was created by constraints *tree has* $n-1$ *edges* and *tree has no cycles*. Formally,

$$STP = \left\{ \mathbf{x} \in [0,1]^{E(G)} : \sum_{e \in E} x_e = n-1, \sum_{uv \in E, u \in X, v \in X} x_{(u,v)} \leq |X| - 1 \text{ for } \emptyset \subset X \subset V \right\}.$$

Suppose we try to characterize the spanning tree by assuming that by constraints *tree has* $n-1$ *edges* and *tree is connected*. The tree is connected can be formulated by saying that for every cut, the sum x_e of edges e in the cut is at least one. Formally,

$$P = \left\{ \mathbf{x} \in [0,1]^{E(G)} : \sum_{e \in E} x_e = n-1, \sum_{uv \in E, u \in X, v \notin X} x_{(u,v)} \geq 1 \text{ for } \emptyset \subset X \subset V \right\}.$$

1. Prove that the spanning tree polytope is a subset of P. That is, $STP \subseteq P$.
2. Show P does NOT have to be the same as the spanning tree polytope. To do this, show that the polytope P does NOT have to be integral (i.e., P contains a vertex that does not have all coordinates integers).

Hint: See the book (in particular exercises).

2: (*Programming spanning tree*)
Implement any minimum spanning tree algorithm and test it on random data. You can pick any algorithm you like. You can use ANY programming language but you have to IMPLEMENT the method yourself (calling a library function `RunKruskal` is not acceptable). Obtain data by randomly generating 10 points in range $[0, 10]^2$ and the cost of every edge is the Euclidean distance in $\mathbb{R}^2$. We consider all 45 edges of K_{10}. Finally, create the plot of of the random points and draw edges picked to the spanning tree. You should provide: Name of the algorithm you implemented and short description of implementation, printout of the source code, pictures of two solutions.
Template is provided for Sage, you do not have to use it.
Time complexity DOES NOT matter.

3: (*Integrality of the shortest path polytope*)
Let $G = (V, E)$ be a directed graph, $s, t \in V$. Let $P \in \mathbb{R}^{|E|}$ be the shortest path polyhedra

defined by equations

$$
\begin{aligned}
\sum_{(s,v)\in E} x_{s,v} - \sum_{(v,s)\in E} x_{v,s} &= 1 \\
-\sum_{(t,v)\in E} x_{t,v} - \sum_{(v,t)\in E} x_{v,t} &= -1 \\
\sum_{(v,w)\in E} x_{v,w} - \sum_{(u,v)\in E} x_{u,v} &= 0 \text{ for all } v \neq s,t \\
x_e &\geq 0 \text{ for all } e \in E
\end{aligned}
$$

1. Argue that P is an integral polyhedra. That is, every vertex of P has all coordinates integers.

   This can be done by showing that for every objective function $\sum_{e\in E} c(e)x_e$, where $c : E \to \mathbb{R}$, if there is an optimal solution, there is one with all coordinates being integers.

2. Find a condition on G that characterizes if P is bounded/unbounded. (Think when it is possible for x_e to go to infinity.)