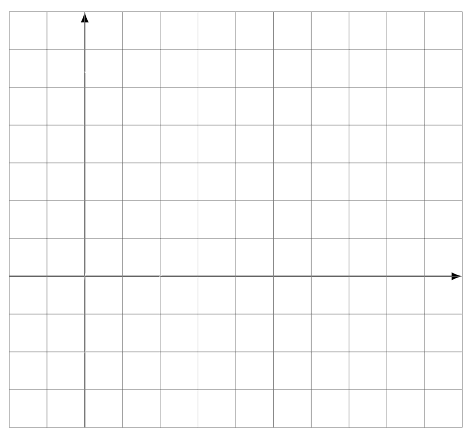
Fall 2015, MATH-566

Chapter 3 - Affine spaces, Radon, Helly

Exercise: Solve the following linear program:

$$(LP)\begin{cases}\text{minimize} & x+y\\ \text{s.t.} & x+2y\leq 14\\ & 3x-y\geq 0\\ & x-y\leq 2,\end{cases}$$



Basic definitions: Suppose we live in $\mathbb{R}^d$ for some $d\in\mathbb{N}$.

- *hyperplane* is $d-1$ dimensional subspace $\{\mathbf{x}\in\mathbb{R}^d : \mathbf{a}\cdot\mathbf{x}=c\}$, where $a\in\mathbb{R}^d$ and $c\in\mathbb{R}$.
- *closed halfspace* is $\{\mathbf{x}\in\mathbb{R}^d : \mathbf{a}\cdot\mathbf{x}\leq c\}$, where $a\in\mathbb{R}^d$ and $c\in\mathbb{R}$.
- $C\subseteq\mathbb{R}^d$ is *convex* if $\forall\mathbf{x},\mathbf{y}\in C,\forall t\in[0,1], t\mathbf{x}+(1-t)\mathbf{y}\in C$. (line between x and y is in C)

Note: Intersection of family of convex sets is a convex set (obvious from the definition)
Convex hull of a set $X=conv(X)$ is the intersection of all convex sets containing X.
Examples: bunch of points, point + line
Claim: $\mathbf{x}\in conv(X)$ iff $\mathbf{x}$ is a convex combination of points from X.
That is:

$$\exists n,\exists\mathbf{x}_1,\dots,\mathbf{x}_n\in X,\exists t_1,\dots,t_n\in[0,\infty),\sum t_i=1, x=\sum t_i\mathbf{x}_i$$

Proof: (as exercise) Hints: consider

$$Y=\left\{\sum t_i\mathbf{x}_i : n\in\mathbb{N},\mathbf{x}_1,\dots,\mathbf{x}_n\in X, t_1,\dots,t_n\in[0,\infty),\sum t_i=1\right\}$$

and show that $Y\subseteq conv(X)$, that $X\subseteq Y$ and Y is convex.

Theorem Carathédory: If $\mathbf{x} \in conv(X)$, then $\mathbf{x}$ is a convex combination of at most $d+1$ points of X.
Proof: HW

Affine subspaces:

Linear subspace: $X \subseteq \mathbb{R}^d$ such that $\mathbf{0} \in X$ and X is closed under addition and multiplication by scalar. (i.e. $\forall \mathbf{u} \in X, \forall \mathbf{v} \in X, \mathbf{u}+\mathbf{v} \in X$ and $\forall \mathbf{u} \in X, \forall a \in \mathbb{R}, a\mathbf{u} \in X$ are both true)

Linear combination: $\mathbf{y}$ is a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ if $\exists t_1, \ldots, t_n \in \mathbb{R}$ such that $\mathbf{Y} = \sum_i t_i \mathbf{x}_i$.
Linear dependence: $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent if x_i that is a linear combination of the rest.
Question: How to check linear dependence?

Affine version - everything is shifter

Affine subspace: "shifted linear space". I.e. $A \subseteq \mathbb{R}^d$ is an affine space if $A = X + \mathbf{v}$, where X is a linear subspace of $\mathbb{R}^d$ and $v \in \mathbb{R}^d$.

Affine combination: $\mathbf{y}$ is an affine combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ if $\exists t_1, \ldots, t_n \in \mathbb{R}$ such that $\mathbf{y} = \sum_i t_i \mathbf{x}_i$ and $\sum_i t_i = 1$.

Affine dependence: $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are affine dependent if exists x_i that is an affine combination of the others.
Question: How to check affine dependence?

Affine hull: Let $X \subseteq \mathbb{R}^d$. Affine hull of X, denote by $aff(X)$, is

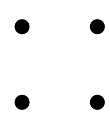
.

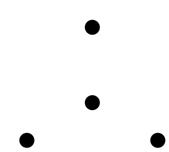
Question: What is the maximum number of affine independent points in $\mathbb{R}^d$?

Question: What is the dimension of the affine hull of k affine independent points?

Radon's theorem: Let A be a set of $d+2$ points in $\mathbb{R}^d$. Then exist $A_1, A_2 \subset A$, such that $A_1 \cap A_2 = \emptyset$ and $conv(A_1) \cap conv(A_2) \neq \emptyset$.

Example for $d = 2$: (find A_1 and A_2)





Helly's theorem: Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$, where $n \geq d+1$. If intersection of every $d+1$ sets is not empty, then $\cap_{i=1}^n C_i \neq \emptyset$.

Question: Show that it is not enough to demand intersection of d of the sets.

Question: How to prove Helly's theorem? (Hint induction on n):

Restatement of Helly's theorem: If $C_1, \ldots, C_n$ are convex and $\cap_{i=1}^n C_i = \emptyset$, then exists at most $d+1$ sets of $C_1, \ldots, C_n$ witnessing that $\cap_{i=1}^n C_i = \emptyset$.

Question: Is infinite version of Helly's theorem true? (that is - intersection of infinitely many convex sets)