Fall 2015, MATH-566

# Chapter 3 - Polytopes

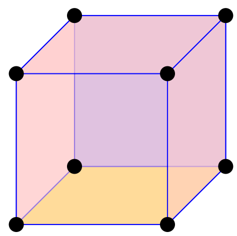
Let $\mathcal{H}$ be a finite set of closed half-spaces and let $V$ be a finite set of points. All in $\mathbb{R}^d$.

*$H$-polyhedron* is $\cap_{H\in\mathcal{H}} H$ (not necessarily bounded)

*$H$-polytope* is a bounded $H$-polyhedron

*$V$-polytope* is $conv(V)$

Example:



Cube is intersection of 6 halfspaces ($H$-polytope) or convex hull of 8 points ($V$-polytope).

Connection to LP:

$$(LP)\begin{cases} \text{minimize} & x+y \\ \text{s.t.} & x+2y \leq 14 \\ & 3x-y \geq 0 \\ & x-y \leq 2, \end{cases}$$

All constraints are half-spaces. The set of *feasible* solutions of $(LP)$ is an $H$-polyhedron. While optimal solution is a vertex of the polyhedron.

*Dimension* of a polyhedron/polytope $P$ is the dimension of $aff(P)$, the affine hull of $P$. (In book dimension defined as $d - rank(A)$ of matrix $A$ in $A\mathbf{x} \leq \mathbf{b}$ - works only for $H$-polytopes/polyhedrons, but result is same.)

**Theorem** Every $H$-polytope is $V$-polytope. Every $V$-polytope is $H$-polytope.

*Face* of a polytope $P$ is:
- $P$ itself - $P \cap h$, where $h$ is a hyperplane s.t. $P$ is entirely in one of the halfspaces determined by $h$.

Note: Every face is a polytope

*Dimension* of a face $F$ of $P$ is the $dim aff(F)$. Dimension of affine space containing $F$.
Special things:

$dim(vertices) = 0$
$dim(\emptyset) = -1$
$dim = 1$ for *edges*
$dim = d - 1$ for *facets*

Note about posets: Take faces as subsets of vertices and order by inclusion.

Polytopes $P$ and $L$ are *combinatorially equivalent* if they have same posets

The poset is actually a *lattice* $\mathcal{L}$ i.e. contains:
- *meet*

$$\forall A, B \in \mathcal{L}, \exists C, C \leq A, C \leq B, \forall K, (K \leq A \wedge K \leq B) \rightarrow (K \leq C)$$

Meet of two faces is their intersection. - *join*

$$\forall A, B \in \mathcal{L}, \exists C, C \geq A, C \geq B, \forall K, (K \geq A \wedge K \geq B) \rightarrow (K \geq C)$$

- All max. chains have the same length
- *atomic* every face is a join of its vertices
- *coatomic* every face is a meet of all facets containing it

$\Rightarrow$ it is enough to know facets or vertices to reconstruct the whole poset
- Not know when poset corresponds to polytope (full characterization)
- For a polytope $P$, its *dual* $P^\star$ is a polytope whose
- Note what happens with facets and vertices in dual.
- Geometric construction of a dual of $P$ in $\mathbb{R}^d$, where $P$ contains the origin $\mathbf{0}$:

$$P^\star = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{y}^T \mathbf{x} \leq 1, \forall \mathbf{x} \in P\}$$

Exercise: construct a geometric dual of square square with side of length 2 and center in the origin. One grid square has size 0.5. What happens with a point?