Fall 2015, MATH-566

Chapter 3 - Simplex and Cyclic polytope

Let P be a polytope. Keep only vertices and edges. Result is a graph.

Theorem: *Steinitz* A finite graph G is isomorphic to a graph of a 3D polytope iff G is planar and vertex 3-connected.

- no simple proof known
- no characterization for higher dimension d known (solution = A from 566)
  it is known that G must be d-connected
- recognition of 4D graph is conjectured to be NP-hard.
- let G be a 3-connected planar graph and P its polytope, then the dual of G is a graph of $P^\star$.

Simplex is a convex hull of affine independent points.

1: Construct simplexes of dimensions 0,1,2 and 3

Simplex is *regular* if all edges have the same length.

2: How to construct regular d-dimensional simplex?

A polytope P is simplicial if every proper face (all but P and $\emptyset$) of P is simplex.

3: Is cube simplicial? Is cross-polytope simplicial?

4: If P is a polytope with all facets being simplexes, is P simplicial?

Points are in *general position* if no j points in $(j-2)$ dimensional affine subspace, for $j = 3, \ldots, d+1$.

5: What does it mean that points in the plane are in general position?

6: If vertices of a polytope P are in general position, is it true that P is simplicial?

Polytope P is *simple* if every j-dimensional face is in $d-j$ facets.

Claim: P is simplicial iff $P^\star$ is simple

crosspolytope is hypercube*.

Recall, that linear program is

$$(LP)\begin{cases}\text{maximize} & (c)^T\mathbf{x}\\ \text{s.t.} & A\mathbf{x}\le\mathbf{b}\end{cases}$$

where $\mathbf{c}\in\mathbb{R}^d$, $A\in\mathbb{R}^{>\times}$, $\mathbf{b}\in\mathbb{R}^m$, $\mathbf{x}\in\mathbb{R}^d$.

Notice $A\mathbf{x}\le\mathbf{b}$ is H-polyhedron. Suppose it is a polytope P. The optimum is in a vertex of P.
Size of (LP) is the number of rows of A, i.e. number of constraints.

Idea of simplex method for solving (LP)

- Find a vertex of $A\mathbf{x}\le\mathbf{b}$
- traverse edges towards the optimum

What is the number of vertices of a polytope that is intersection of m half-spaces?

7: What is the number of facets and vertices of a d-dimensional hypercube?

Conjecture Hirsh: Graph of a polytope of dimension d with n facets has diameter $n-d$.

- would give linear number of steps for simplex method
- best upper bond is $2n^{\log(d+1)}$
- conjecture is not true, but maybe the diameter is still linear

Cyclic polytope (something with few vertices and many facets - dual is scary for linear programming)
Moment curve is $\gamma=\{(t,t^2,t^3,\dots,t^d):t\in\mathbb{R}\}\subseteq\mathbb{R}^d$

8: What is the moment curve for $d=2$?

Lemma: Hyperplane H and a moment curve γ intersect in at most d points. If they intersect in d points, then none of them is *tangent*.

9: Show that every d points on the moment curve are affine independent

Moment curve is an example of points in general position.

Definition: Let $V\subset\gamma$ and $|V|$ finite. Then $conv(v)$ is *cyclic polytope*.

10: What is the number of facets of cyclic polytope? (roughly)

Theorem: Cyclic polytope maximizes the number of faces of all dimensions among all polytopes with n vertices.