

## Chapter 3 - Interpretation of Duality and Duality theorem

Dualization for everyone:

$$A \in \mathbb{R}^{m \times n}, \mathbf{c} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$$

	primal	dual
variables	$x_1, \dots, x_n$	$y_1, \dots, y_m$
matrix	$A$	$A^T$
right hand	$\mathbf{b}$	$\mathbf{c}$
objective	$\max \mathbf{c}^T \mathbf{x}$	$\min \mathbf{b}^T \mathbf{y}$
constraint	$i$ th constrain $\leq$	$y_i \geq 0$
	$i$ th constrain $\geq$	$y_i \leq 0$
	$i$ th constrain $=$	$y_i \in \mathbb{R}$
	$x_i \geq 0$	$i$ th constrain $\geq$
	$x_i \leq 0$	$i$ th constrain $\leq$
	$x_i \in \mathbb{R}$	$i$ th constrain $=$

**Diet problem:** How much apricots ( $x_1$ ), bananas ( $x_2$ ) and cucumbers ( $x_3$ ) one has to eat to get enough of Vit A, B, C? Minimize cost.

Need to know: % of daily value and cost:

	A	C	K	\$	ammount
apricots	60	26	6	1.53	155g
bananas	3	33	1	0.37	225g
cucumbers	2	7	12	0.18	133g

**1:** Write Linear Program ( $P$ ) solving the diet problem and write also its dual ( $D$ )

**Solution:**

$$(P) \begin{cases} \text{minimize} & 1.53x_1 + 0.37x_2 + 0.18x_3 \\ \text{s.t.} & 60x_1 + 3x_2 + 2x_3 \geq 100 \\ & 26x_1 + 33x_2 + 7x_3 \geq 100 \\ & 6x_1 + 1x_2 + 12x_3 \geq 100 \\ & x_1, x_2, x_3 \geq 0 \end{cases} \quad (D) \begin{cases} \text{maximize} & 100y_1 + 100y_2 + 100y_3 \\ \text{s.t.} & 60y_1 + 26y_2 + 6y_3 \leq 1.53 \\ & 3y_1 + 33y_2 + 1y_3 \leq 0.37 \\ & 2y_1 + 7y_2 + 12y_3 \leq 0.18 \end{cases}$$

Solution for ( $P$ ) is  $\mathbf{x} = (1.4, 0.3, 7.6)$  and the cost is \$3.62.

and for ( $D$ ) is  $\mathbf{y} = (0.0209, 0.009122, 0.006191)$  and the cost is \$3.62.

**2:** What are units of  $y_i$  in ( $D$ )? (Hint: inequalities need to make sense in units.)

**Solution:** Units of  $y_i$  are  $\frac{\$}{1\% \text{ of daily intake of vitamin}}$ .

**3:** Imagine you want to create a multivitamine pills ACK. What is the maximum price of one ACK pull if it has to deliver 100% of recommended daily value of vitamins A,C, and K and it must beat any fruit and vegetable in terms of price? (*If you don't manage to beat fruit and vegetable, nobody will buy your fancy ACK pill*) (Compute your price as a combination of prices of each of the vitamins.)

**Solution:** Same as the dual program ( $D$ ).

$$(D) \begin{cases} \text{maximize} & 100y_1 + 100y_2 + 100y_3 \\ \text{s.t.} & 60y_1 + 26y_2 + 6y_3 \leq 1.53 \\ & 3y_1 + 33y_2 + 1y_3 \leq 0.37 \\ & 2y_1 + 7y_2 + 12y_3 \leq 0.18 \end{cases}$$

### Duality Theorem

For the linear programs

$$\text{maximize } \mathbf{c}^T \mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \quad (P)$$

and

$$\text{minimize } \mathbf{b}^T \mathbf{y} \text{ subject to } A^T \mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \quad (D)$$

exactly one of the following possibilities occurs:

1. Neither ( $P$ ) nor ( $D$ ) has a feasible solution.
2. ( $P$ ) is unbounded and ( $D$ ) has no feasible solution.
3. ( $P$ ) has no feasible solution and ( $D$ ) is unbounded.
4. Both ( $P$ ) and ( $D$ ) have a feasible solution. Then both have an optimal solution, and if  $\mathbf{x}^*$  is an optimal solution of ( $P$ ) and  $\mathbf{y}^*$  is an optimal solution of ( $D$ ), then

$$\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*.$$

That is, the maximum of ( $P$ ) equals the minimum of ( $D$ ).

*Next goal is to prove the duality theorem.*