

### Chapter 3 - Farkas Lemma and proof of duality

**Farkas Lemma:** Let  $A \in \mathbb{R}^{m \times n}$  and  $\mathbf{b} \in \mathbb{R}^m$ . Exactly one of the following holds

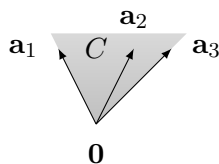
- $\exists \mathbf{x} \in \mathbb{R}^n$  such that  $A\mathbf{x} = \mathbf{b}$  and  $\mathbf{x} \geq \mathbf{0}$
- $\exists \mathbf{y} \in \mathbb{R}^m$  such that  $\mathbf{y}^T A \geq \mathbf{0}^T$  and  $\mathbf{y}^T \mathbf{b} < 0$

**1:** Is it possible to satisfy both conditions at the same time?

A (convex) **cone** is a set  $C \in \mathbb{R}^d$  for which  $\mathbf{x}, \mathbf{y} \in C$  and  $a, b \geq 0$  implies  $a\mathbf{x} + b\mathbf{y} \in C$ .

A cone  $C$  generated by  $X = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subseteq \mathbb{R}^d$  are all linear combinations of vectors in  $X$  with nonnegative coefficients

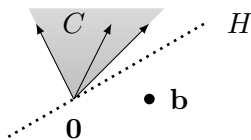
$$C = \{t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \dots + t_n\mathbf{a}_n : t_i \geq 0\} \subseteq \mathbb{R}^d$$



Convex cone can be defined for any generating set  $X$ . If  $X$  is finite, then  $C$  is closed.

**Geometric of Farkas Lemma:** Let  $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{b} \in \mathbb{R}^m$ . Let  $C$  be the convex cone generated by  $\mathbf{a}_i$ s. Exactly one of the following holds

- $\mathbf{b} \in C$
- exists hyperplane  $H$  such that  $\mathbf{0} \in H$  and  $H$  strictly separates  $\mathbf{b}$  from  $C$ . That is  $H = \{\mathbf{x} : \mathbf{h}^T \mathbf{x} = 0\}$  and  $\forall i, \mathbf{h}^T \mathbf{a}_i \geq 0$  and  $\mathbf{h}^T \mathbf{b} < 0$ .



**2:** Prove Farkas lemma using separation theorem. (What the separation gives?)

Reformulations of Farkas lemma:

- $A\mathbf{x} = \mathbf{b}$  has a non-negative solution iff  $\forall \mathbf{y} \in \mathbb{R}^m$  with  $\mathbf{y}^T A \geq \mathbf{0}^T$  also  $\mathbf{y}^T \mathbf{b} \geq 0$ .
- $A\mathbf{x} \leq \mathbf{b}$  has a non-negative solution iff  $\forall \mathbf{y} \in \mathbb{R}^m, \mathbf{y} \geq \mathbf{0}$  with  $\mathbf{y}^T A \geq \mathbf{0}^T$  also satisfies  $\mathbf{y}^T \mathbf{b} \geq 0$ .
- $A\mathbf{x} \leq \mathbf{b}$  has a solution iff  $\forall \mathbf{y} \in \mathbb{R}^m, \mathbf{y} \geq \mathbf{0}$  with  $\mathbf{y}^T A = \mathbf{0}^T$  also satisfies  $\mathbf{y}^T \mathbf{b} \geq 0$ .

Lets have linear programs

$$\text{maximize } \mathbf{c}^T \mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0} \quad (P)$$

$$\text{minimize } \mathbf{b}^T \mathbf{y} \text{ subject to } A^T \mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0} \quad (D)$$

Lemma (Weak Duality): Let  $\mathbf{x}$  and  $\mathbf{y}$  be feasible solutions of (P) and (D). Then

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$$

**3:** Prove the weak duality

**Proof of the duality theorem point 4. from the Farkas lemma.** ( $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$ .)

Let  $\mathbf{x}^*$  be optimal solution. Let  $\gamma = \mathbf{c}^T \mathbf{x}^*$ .

**4:** Are there solutions to  $A\mathbf{x} \leq \mathbf{b}$  and  $\mathbf{c}^T \mathbf{x} \geq \gamma$ ?

**5:** Are there solutions to  $A\mathbf{x} \leq \mathbf{b}$  and  $\mathbf{c}^T \mathbf{x} \geq \gamma + \varepsilon$ , where  $\varepsilon > 0$ ?

Let  $\hat{A} = \begin{pmatrix} A \\ -\mathbf{c}^T \end{pmatrix}$  and  $\hat{\mathbf{b}}_\varepsilon = \begin{pmatrix} \mathbf{b} \\ -\gamma - \varepsilon \end{pmatrix}$ .

**6:** Apply Farkas Lemma on  $\hat{A}\mathbf{x} \leq \hat{\mathbf{b}}_\varepsilon$  (which version?, write  $\hat{\mathbf{y}}$  from FL as  $(\mathbf{u}, z) \in \mathbb{R}^{m+1}$  ?)

**7:** What happens if  $z = 0$ ? (Hint: Use Farkas lemma again with  $\varepsilon = 0$ .)