Fall 2015, MATH-566

Linear Programming Algorithms - Interior point methods

Source: Chapter 11 of Convex Optimization, Stephen Boyd and Lieven Vandenberghe

Let

$$(P)\begin{cases}\text{minimize} & f(\mathbf{x})\\ \text{subject to} & g_i(\mathbf{x}) \le 0 \text{ for } i = 1,\dots,m\end{cases}$$

where f, g are convex, twice continuously differentiable and optimal solution $\mathbf{x}^\star$ exists. Moreover, let (P) be *superconsisten*, $\exists\mathbf{x}, \forall i, g_i(\mathbf{x}) < 0$.

(the setup covers linear, quadratic, geometric, semidefinite, ... programming).

Idea: Change the (P) to a problem without constraints.

Let

$$(P') = \text{minimize } f(\mathbf{x}) + \sum_{i=1}^{m} I(g_i(\mathbf{x})),$$

where I is an indicator function

$$I(u) = \begin{cases} 0 & \text{if } u \le 0\\ +\infty & \text{if } u > 0.\end{cases}$$

1: What is the optimal solution to (P')?

2: Can you solve (P') by methods from calculus?

Use approximation of $I(u) \approx -c\log(-u)$, where $c > 0$.

3: Sketch $I(u)$ and its approximations. Is the approximation better when c is large or small?

For $t > 0$, we consider a smooth unconstrained approximation of (P')

$$\text{minimize } f(\mathbf{x}) - \frac{1}{t}\sum_{i=1}^{m} \log(-g_i(\mathbf{x})).$$

Define *logarithmic barrier function*

$$\Phi(\mathbf{x}) = -\sum_{i=1}^{m} \log(-g_i(\mathbf{x})),$$

for all $\mathbf{x}$ where $g_i(\mathbf{x}) < 0$ (interior of feasible solutions)

Analytic center of the set $S = \{\mathbf{x} : g_i(\mathbf{x}) \leq 0\} \subseteq \mathbb{R}^n$ is $\mathbf{x}^\star$ minimizing $\Phi(\mathbf{x})$ over all $\mathbf{x} \in S$.

4: Find the analytic center of a square in $\mathbf{R}^2$ defined by equations

$$x_1 \geq 0, x_2 \geq 0, x_1 \leq 1, x_2 \leq 1$$

5: Find the analytic center of a square in $\mathbf{R}^2$ defined by equations

$$x_1 \geq 0, x_2 \geq 0, (1-x_1)^3 \geq 0, (1-x_2)^3 \geq 0.$$

Notice it is possible to define center even if functions are not convex everywhere and the center depends on the functions.

For $t > 0$ define $\mathbf{x}^\star(t)$ as the optimal solution of

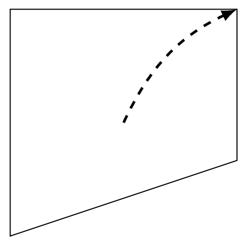
$$(P_t) = \text{ minimize } tf(\mathbf{x}) + \Phi(\mathbf{x}).$$

(assume that the optimal solution is unique)

Central path is $\{\mathbf{x}^\star(t) : t \geq 0\}$.

Interior point method idea: Start in the analytical center and follow the central path.

In iterations increase t and recompute the new optimum by Newton's method.



There exists a notion of dual program (D) for (P), (based on Karush-Kuhn-Tucker theorem). It gives solutions to the dual $\mathbf{y}^\star(t)$ such that

$$f(\mathbf{x}^\star(t)) - h(\mathbf{y}^\star(t)) \leq \frac{m}{t}.$$

Hence the central path converges to $\mathbf{x}^\star$ for (P).

6: Let (LP) = mininize $\mathbf{c}^T\mathbf{x}$ s.t. $A\mathbf{x} \leq \mathbf{b}$, where $A \in \mathbb{R}^{m\times n}$. Write $\Phi(\mathbf{x})$. Use tools from calculate conditions for the corresponding central path. What can you say about the Hessain?

In the end we obtained

$$t\mathbf{c} = -\nabla\Phi(\mathbf{x}^\star(t)).$$

Since $\nabla\Phi(\mathbf{x})$ is perpendicular to the level curve $\{\mathbf{x} : \Phi(\mathbf{x}) = \Phi(\mathbf{x}^\star(t)\}$, plane $\mathbf{c}^T\mathbf{x} = \mathbf{c}^T\mathbf{x}^\star(t)$ is a tangent of the level curve for Φ.