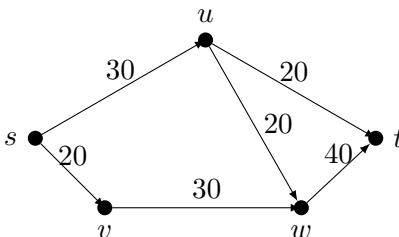


## Network flows - first introduction

*Original application (product of cold war)*

Suppose country  $X$  enters into a war. How quickly can  $X$  move tank from storage  $s$  to the target  $t$  (battle ground)? The tanks are moved on a railroad. Every link gives the capacity how many tank a day can be transported.



**1:** How many tanks per day can be delivered to the battleground? Is the solution unique?

**Problem:** (Directed) graph  $G$ , source  $s$ , sink  $t$ , capacities  $u : E(G) \rightarrow \mathbb{R}_+$ .

**Network** is  $(G, u, s, t)$ .

Input: Network  $(G, u, s, t)$

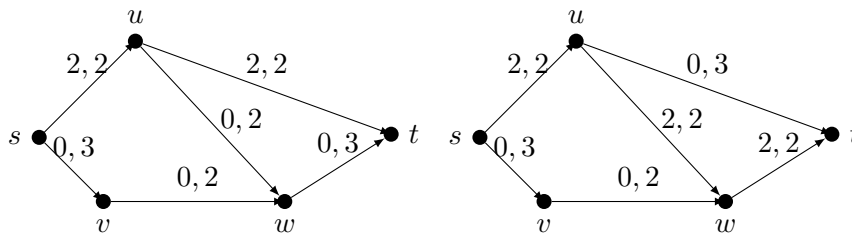
Output:  $s$ - $t$ -flow of maximum value

$s$ - $t$ -**flow**  $f$  is a function  $f : E(G) \rightarrow \mathbb{R}_+$ .

**Value** of  $f$  is  $\sum_{su \in E} f(su) - \sum_{us \in E} f(us)$  i.e. leaving – entering to  $s$ .

**2:** How  $f$  looks around one vertex of the network? (what must  $f$  satisfy?)

**3:** How to improve these flows to be maximum? Description on edges are values of  $f, u$ .



**4:** After the improvement, how do you argue that nobody can further improve?

Let  $A \subset V(G)$  such that  $s \in A$  and  $t \notin A$ . Use  $\delta^+(A)$  to denote set of edges  $uv$ , where  $u \in A$  and  $v \notin A$  (edges leaving  $A$ ). Use  $\delta^-(A)$  to denote set of edges  $uv$ , where  $u \notin A$  and  $v \in A$  (edges entering  $A$ ).

**Capacity** of  $s$ - $t$ -cut  $A$  is  $\sum_{e \in \delta^+(A)} u(e)$ .

**5:** Prove that for  $A$  and any flow  $f$  holds

(a)  $\text{value}(f) = \sum_{e \in \delta^+(A)} f(e) - \sum_{e \in \delta^-(A)} f(e)$

(b)  $\text{value}(f) \leq \sum_{e \in \delta^+(A)} u(e)$

This proves the *obvious* observation that maximum flow cannot exceed capacity of minimum cut.

Notice in 3. we were improving flow by reducing the flow on  $uw$ . We “sent flow in the opposite direction”.

For a digraph  $G$ , define  $\overleftrightarrow{G}$  by adding for every edge  $e$  also its **reverse**  $\overleftarrow{e}$ .

For  $f$  and  $u$  define **residual capacities**  $u_f : E(\overleftrightarrow{G}) \rightarrow \mathbb{R}_+$

$$u_f(e) = u(e) - f(e)$$

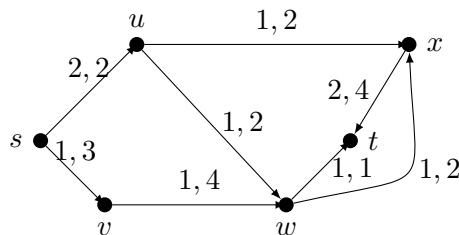
$$u_f(\overleftarrow{e}) = f(e)$$

Residual capacities ... how much extra we can send in each direction.

**Residual graph**  $G_f$  is obtained from  $\overleftrightarrow{G}$  by removing edges  $e \in E(\overleftrightarrow{G})$  with  $u_f(e) = 0$ .

**Augmenting path** is an  $s$ - $t$  path in  $G_f$ .

**6:** Construct the residual graph for



and find an augmenting path and increase the flow using the augmenting path.

**7:** How to update (to **augment** the flow  $f$  using augmenting path in  $\overleftrightarrow{G}$ ?