Fall 2015, MATH-566

Minimum Cost Flow

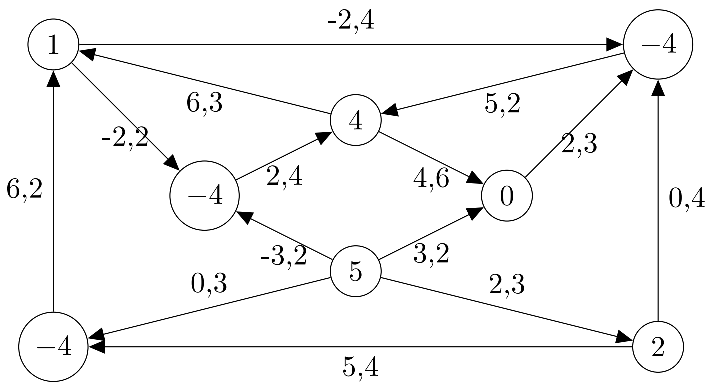
Problem: There are n coal mines and m power plants. Power plants have demands, coal mines supply coal. How to transport coal in order to satisfy the demands and minimize cost of transportation.

Let $G = (V, E)$ be a directed graph, $u : E \to \mathbb{R}_+$ be capacities on edges and $c : E \to \mathbb{R}$ be costs for every edge.

Let $b : V \to \mathbb{R}$ with $\sum_v b(v) = 0$ be a *supply demand* function.

b-flow is $f : E \to \mathbb{R}_+$ such that $f(e) \le u(e)$ and $\sum_{e\in\delta^+(v)} f(e) - \sum_{e\in\delta^-(v)} f(e) = b(v)$.

1: Find a b-flow (that minimizes $\sum_e c(e)f(e)$) in the following network: (b is in every vertex, edges have c, u).



If $b(v) > 0$, then b is *supply*, if $b(v) < 0$, then b is demand. Like flows but multiple sources and sinks.

Minimum Cost Flow Problem: find a b-flow f that minimizes $c(f) = \sum_e c(e)f(e)$.

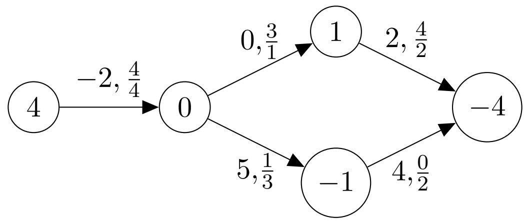
2: Show that b-flow f exists iff

$$\sum_{e\in\delta^+(X)} u(e) \ge \sum_{v\in X} b(v) \text{ for all } X \subseteq V(G).$$

(That is, there is always enough capacity to take excessive flow out of X.)

Consequence: It is possible to detect no solution case.

3: Let f and f' be two b-flows. Consider their *difference* $f - f'$ and show that it is a circulation.
Try on example first: Edge labels are $c, \frac{f}{f'}$. Compute $c(f)$, $c(f')$, find what is the difference.

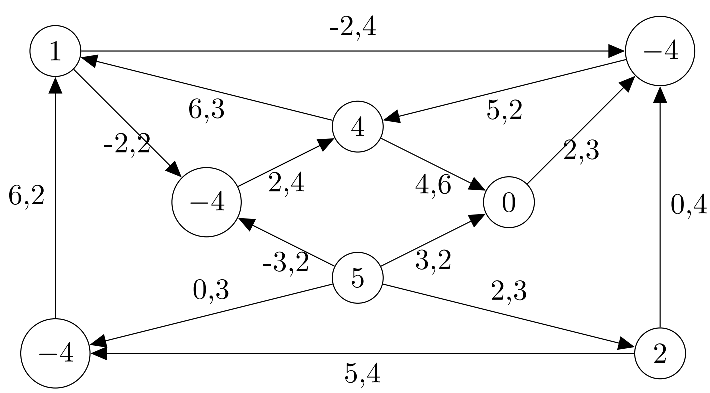


Algorithm Minimum Cost Flow:

1. f be any b-flow
2. while exists negative cost cycle C in residual graph
3. pick C of minimum mean cost $= \frac{\sum_{e \in C} c(e)}{|C|}$.
4. augment on C

Minimum mean cost cycle gives polynomial time $O(m^2 n^2 \log n)$ (without - same problem as Ford-Fulkerson).

4: Run the algorithm on



5: Show that the algorithm is correct when it finishes. That is, f is an optimal b-flow iff it has no negative cycle.

6: How to find minimum mean cycle?

Next time: Minimum Mean Cycle and Integer Programming