Name: ....

MATH-165 **Puzzle Collection 4**

2016 Nov 18 12:10pm-1:00pm

23 12:10pm–Wumaier	24 12:10pm–Njus
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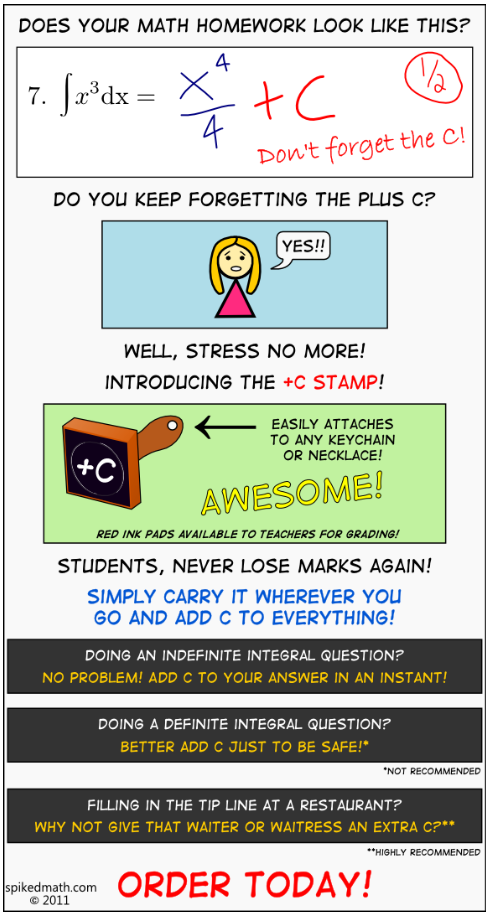
This puzzle collection is closed book and closed notes. NO calculators are allowed for these puzzles. For full credit show all of your work (legibly!). Every puzzle is worth 10 points (a total of 50 points).
If you do not mark your section correctly, you will get -2 points.

Good luck!

Puzzle 1	Puzzle 2	Puzzle 3	Puzzle 4	Puzzle 5
/10	/10	/10	/10	/10

Total score
/50



This version contains answers!

These are not answers you should write on the exam but answers you could study and they should help you understand HOW the solution was obtained.

1: Use the Fundamental Theorem of Calculus to compute C and $f(4)$, where

$$\int_0^x f(t)\,dt = x\cos(\pi x) + 1 + Ce^x.$$

We have two things to do so lets start with the easier one: C. It would be cool to know the value of the left-hand side for some x and then we could solve for C. Luckily, there is such x if we use

$$\int_a^a g(y)\,dy = 0$$

an here does not matter what a or g is. So in our case, we can use $x = 0$ and we get

$$0 = \int_0^0 f(t)\,dt = 0\cos(\pi 0) + 1 + Ce^0.$$

Recall that $e^0 = 1$ so we have equation

$$0 = 1 + C$$

Hence $C = -1$ and we can write

$$\int_0^x f(t)\,dt = x\cos(\pi x) + 1 - e^x.$$

Time to compute $f(x)$. Now fundamental theorem of calculus is saying $\frac{d}{dx}\left(\int_a^x g(y)\,dy\right) = g(x)$. So we can compute

$$\begin{aligned} \frac{d}{dx}\left[\int_0^x f(t)\,dt\right] &= \frac{d}{dx}\left[x\cos(\pi x) + 1 - e^x\right] \\ f(x) &= \cos(\pi x) - x\sin(\pi x)\pi - e^x \end{aligned}$$

No we just evaluate for $x = 4$. Recall that $\sin(4\pi) = 0$ and $\cos(4\pi) = 1$ and we get

$$\begin{aligned} f(4) &= \cos(4\pi) - 4\sin(4\pi)\pi - e^4 \\ &= 1 - e^4 \end{aligned}$$

Notice that we actually needed C to get the answer for $f(4)$. If we did not compute C first, we would obtain $f(4) = 1 + C \cdot e^4$.

The final answer is

$$f(4) = 1 - e^4 \qquad\qquad C = -1$$

$f(4) =$ _______________ $C =$ _______________

2: Evaluate

$$\int_1^{e^5} \frac{2}{t(4+\ln t)^{3/2}}\, dt$$

Your answer must be in simplest form.

This looks like something for a substitution. A bright eye can see that $\frac{d}{dt}(\ln t) = \frac{1}{t}$ and both of these are in the fraction if it is written as

$$\int_1^{e^5} \frac{2}{t(4+\ln t)^{3/2}}\, dt = \int_1^{e^5} \frac{2}{(4+\ln t)^{3/2}} \cdot \frac{1}{t}\, dt$$

So we try to use substitution $u = \ln t$. This gives $du = \frac{1}{t}\, dt$. Also critical are the bounds of integration. They could also help with guessing the substitution. Look, both $\ln 1 = 0$ and $\ln e^5 = 5$ were real easy to evaluate. The outcome of substitution is

$$\int_1^{e^5} \frac{2}{(4+\ln t)^{3/2}} \cdot \frac{1}{t}\, dt = \int_0^5 \frac{2}{(4+u)^{3/2}}\, du.$$

Now we could either make the antiderivative right away or we could do one more substitution. Lets make one more: $z = 4+u$ and $dz = du$. Very easy! And do not forget to change the bounds of integration.

$$\int_0^5 \frac{2}{(4+u)^{3/2}}\, du = \int_4^9 \frac{2}{z^{3/2}}\, dz$$

Notice we could take a substitution $z = 4 + \ln t$m, which gives $dz = \frac{1}{t}\, dt$ and do it all in one step.

Now we have a fraction to integrate. We could get a fraction as

$$\ln(x)' = \frac{1}{x} \qquad x^k = \frac{1}{x^{-k}} \qquad (\arctan x)' = \frac{1}{1+x^2}$$

$\frac{2}{z^{3/2}}$ does not seem to match ln or arctan nicely so we go with the middle one (notice that the middle is NOT a derivative yet unlike the sides.)

$$\int_4^9 \frac{2}{z^{3/2}}\, dz = \int_4^9 2 \cdot z^{-3/2}\, du$$

Now it is time to take the antiderivative as a polynomial and evaluate

$$\int_4^9 2 \cdot z^{-3/2}\, du = \left[2 \cdot (-2) z^{-1/2}\right]_4^9 = \left[\frac{-4}{\sqrt{z}}\right]_4^9 = \frac{-4}{3} + \frac{4}{2} = \frac{2}{3}$$

The answer is

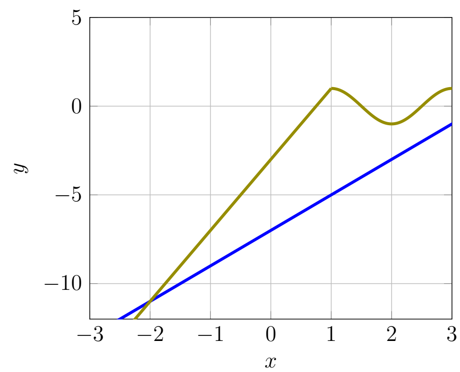
$$\int_1^{e^5} \frac{2}{t(4+\ln t)^{3/2}}\, dt = \frac{2}{3}.$$

$$\int_1^{e^5} \frac{2}{t(4+\ln t)^{3/2}}\, dt = \underline{\hspace{4cm}}$$

3: A function g is defined by

$$g(x) = \begin{cases} 4x-3 & x < 1 \\ -\cos(\pi x) & x \geq 1 \end{cases}$$

and f is defined as $f(x) = 2x - 7$. Compute the area between the curves f and g for $-2 \leq x \leq 2$.



This looks like simple area computation. We use that the are between curves f and g on interval $[a, b]$ can be computed as $\int_a^b g(x) - f(x)\, dx$, where $g(x) \geq f(x)$ for all $x \in [a, b]$.

Notice that indeed $g(x) \geq f(x)$ for all $-2 \leq x \leq 2$. If it was not the case, we would swap f and g. So we need to compute

$$\int_{-2}^{2} g(x) - f(x)\, dx$$

It is easy to say what is $f(x)$ but $g(x)$ is quite bad. Luckily, we can split the integral and make it easy

$$\int_{-2}^{2} g(x) - f(x)\, dx = \int_{-2}^{1} g(x) - f(x)\, dx + \int_{1}^{2} g(x) - f(x)\, dx$$

Alternatively, we could also do

$$\int_{-2}^{2} g(x) - f(x)\, dx = \int_{-2}^{2} g(x)\, dx - \int_{-2}^{2} f(x)\, dx = \int_{-2}^{1} g(x)\, dx + \int_{1}^{2} g(x)\, dx - \int_{-2}^{2} f(x)\, dx$$

We will go with the second option as it is easier to compute, but sometimes the first one might be necessary.

By replacing g and f and taking the antiderivatives we get

$$\int_{-2}^{1} g(x)\, dx + \int_{1}^{2} g(x)\, dx - \int_{-2}^{2} f(x)\, dx = \int_{-2}^{1} 4x - 3\, dx \int_{1}^{2} -\cos(\pi x)\, dx - \int_{-2}^{2} 2x - 7\, dx$$
$$= \left[2x^2 - 3x\right]_{-2}^{1} + \left[-\sin(\pi x)\right]_{1}^{2} - \left[x^2 - 7x\right]_{-2}^{2}$$
$$= (-1) - (14) + 0 - 0 - (-10 - (4 + 14)) = 13$$

A way to check the solution is that the area is ≥ 0. Negative area would be bad. Hence

$$\text{Area} = 13$$

Area = ______________

4: Find y given $y' = (y^2+1)\cdot\cos(x)$ and $y(\pi/2) = 0$.

Here we have a separable differential equation. We can try separate x and y and take antiderivatives and solve for y. Note that $y' = \frac{dy}{dx}$.

$$\begin{aligned}
\frac{dy}{dx} &= (y^2+1)\cdot\cos(x) \\
\frac{1}{y^2+1}\frac{dy}{dx} &= \cos(x) \\
\int \frac{1}{y^2+1}\,dy &= \int \cos(x)\,dx \\
\arctan(y) &= \sin(x) + C \\
y &= \tan(\sin(x) + C)
\end{aligned}$$

Now we need to compute what is C. We compute it from $\arctan(y) = \sin(x) + C$ using $y = 0$ and $x = \pi/2$.

$$\begin{aligned}
\arctan(0) &= \sin(\frac{pi}{2}) + C \\
0 &= 1 + C \\
C &= -1
\end{aligned}$$

Hence the answer is

$$y = \tan(\sin(x) - 1).$$

$y =$______________

5: Compute

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\arctan x}\right)$$

The limit looks like $\frac{1}{0}-\frac{1}{0}$. So splitting it into two limits is a bad idea. Also, we cannot use l'Hospital yet since it works only for $\frac{0}{0}$ or $\frac{\infty}{\infty}$. So we try to put the terms together into one fraction.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\arctan x}\right)=\lim_{x\to 0}\left(\frac{\arctan x}{x\cdot\arctan x}-\frac{x}{x\cdot\arctan x}\right)=\lim_{x\to 0}\frac{x-\arctan x}{x\cdot\arctan x}$$

Now as $x\to 0$, the limit looks like $\frac{0}{0}$. So we can use l'Hospital. Do not forget that we take the derivative of numerator and denominator separately. We do NOT take the derivative of the whole fraction.

$$\lim_{x\to 0}\frac{x-\arctan x}{x\cdot\arctan x}=\lim_{x\to 0}\frac{1-\frac{1}{1+x^2}}{\arctan x+\frac{x}{1+x^2}}$$

We again got fraction that looks like $\frac{0}{0}$. So we can try to use l'Hopital again!

$$\lim_{x\to 0}\frac{1-\frac{1}{1+x^2}}{\arctan x+\frac{x}{1+x^2}}=\lim_{x\to 0}\frac{0+\frac{2x}{(1+x^2)^2}}{\frac{1}{1+x^2}+\frac{1}{1+x^2}-x\cdot\frac{2x}{(1+x)^2}}$$

Although the fraction looks horrible, it behaves like $\frac{0}{2}=0$. Hence

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\arctan x}\right)=0.$$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\arctan x}\right)=\underline{\hspace{4cm}}$$