1. Find $\lim_{t \to 0} \dfrac{e^{2t} - 1 - 2t}{1 - \cos(3t)}$. $\leftarrow$ L'Hospital

$$\lim_{t \to 0} \frac{e^{2t} - 1 - 2t}{1 - \cos(3t)} \longrightarrow \frac{e^{0} - 1 - 2\cdot 0}{1 - \cos(0)} = \frac{0}{0}$$

$$\overset{\text{L.H.}}{=} \lim_{t \to 0} \frac{2e^{2t} - 2}{3\sin(3t)} \longrightarrow \frac{2e^{0} - 2}{3\sin(0)} = \frac{0}{0}$$

$$\overset{\text{L.H.}}{=} \lim_{t \to 0} \frac{4e^{2t}}{9\cos(3t)} = \frac{4e^{0}}{9\cos(0)} = \boxed{\frac{4}{9}}$$

score

2. Find $\int_0^{\pi/4} (2 - \sec^2(\theta))^{1/2} \underbrace{\sec^2(\theta)\, d\theta}$. (Hint: recall that $\tan^2(\theta) + 1 = \sec^2(\theta)$.)

Suggests substitution of $u = \tan\theta$

$$\int_0^{\pi/4} \left(2 - (\tan^2\theta + 1)\right)^{1/2} \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} (1 - \tan^2\theta)^{1/2} \sec^2\theta \, d\theta$$

$u = \tan\theta$

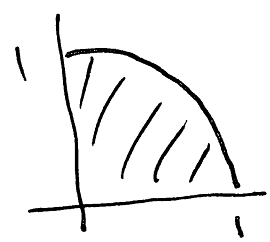
$du = \sec^2\theta \, d\theta$

$\tan(0) = 0$

$\tan(\pi/4) = 1$

$$= \int_0^1 (1-u^2)^{1/2} \, du$$

$$= \boxed{\frac{1}{4}\pi}$$



$\frac{1}{4}$ of unit circle has area $\frac{1}{4}\pi$

score

$a = 32$

3. (a) Find the linearization to $f(x) = x^{1/5}$ at $x = 32$.

$L(x) = f(a) + f'(a)(x-a)$

$\boxed{= 2 + \frac{1}{80}(x-32)}$

$f(32) = 32^{1/5} = 2$

$f'(x) = \frac{1}{5}x^{-4/5} = \frac{1}{5(x^{1/5})^4}$

$f'(32) = \frac{1}{5 \cdot 2^4} = \frac{1}{80}$

(b) Use the linearization from (a) to estimate $(34)^{1/5}$.

$34^{1/5} = f(34) \approx L(34) = 2 + \frac{1}{80}(34-32)$

$= 2 + \frac{2}{80}$

$= 2 + \frac{1}{40}$

$\boxed{= 2.025}$

score

4. Find $\int \frac{dy}{y^{1/3}(y^{2/3}+1)}$.

$u = y^{2/3} + 1$

$du = \frac{2}{3} y^{-1/3} dy$ or $\frac{3}{2} du = \frac{1}{y^{1/3}} dy$

$\int \frac{1}{y^{2/3}+1} \cdot \frac{1}{y^{1/3}} dy = \int \frac{1}{u} \cdot \frac{3}{2} du$

$= \frac{3}{2} \ln(u) + C$

$\boxed{= \frac{3}{2} \ln(y^{2/3}+1) + C}$

score

5. Determine the absolute max and min for $f(x) = x - |x^2 - 5x - 6|$ for $0 \leqslant x \leqslant 7$.

$$x^2 - 5x - 6 = (x-6)(x+1)$$

absolute value changes behavior at -1 & 6

$$f(x) = x - |x^2 - 5x - 6| = \begin{cases} x + (x^2 - 5x - 6) & 0 \le x \le 6 \\ x - (x^2 - 5x - 6) & 6 \le x \le 7 \end{cases}$$

$$= \begin{cases} x^2 - 4x - 6 & 0 \le x \le 6 \\ -x^2 + 6x + 6 & 6 \le x \le 7 \end{cases}$$

$$f'(x) = \begin{cases} 2x - 4 & 0 \le x < 6 \\ -2x + 6 & 6 < x \le 7 \end{cases}$$

$2x - 4 = 0$ at $x = 2$ which is in $[0, 6]$ so 2 a critical pt

$-2x + 6 = 0$ at $x = 3$ which is not in $[6, 7]$ so ignore

@ $x = 6$ derivatives don't match so no derivative, i.e., 6 a critical point

Critical pts

bdy $\{ 0, 7$

$f' = 0$ $\{ 2$

f' DNE $\{ 6$

$f(0) = -6$

$f(7) = -1$

$f(2) = -10$ ← abs. min

$f(6) = 6$ ← abs. max

score

6. Suppose the weight of a cloud in tons after t hours, denoted $W(t)$, decreases proportionally to its surface area and that this process speeds up over time until the cloud is completely gone. We can approximate this behavior with the differential equation $W' = -\frac{5}{3}tW^{2/3}$. Given that the cloud initially weights 1000 tons, how many hours until the cloud is completely gone?

$$\frac{dW}{dt} = W' = -\frac{5}{3}tW^{2/3}$$

$$\frac{dW}{W^{2/3}} = -\frac{5}{3}t\,dt$$

$$\int \frac{dW}{W^{2/3}} = \int -\frac{5}{3}t\,dt$$

$$\int W^{-2/3}dW$$

$$3W^{1/3} = -\frac{5}{6}t^2 + C$$

@ $t=0$, $W=1000$

$$3 \cdot 1000^{1/3} = 0 + C$$

$$30 = C$$

$$3W^{1/3} = -\frac{5}{6}t^2 + 30$$

need to find t so that $W=0$

$$-\frac{5}{6}t^2 + 30 = 0$$

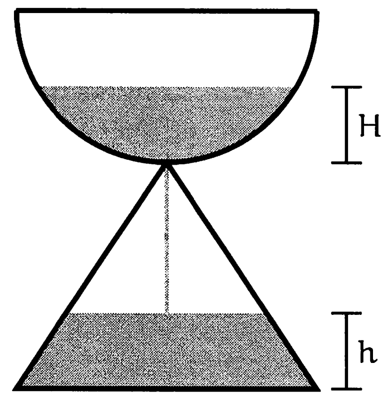
$$\frac{5}{6}t^2 = 30$$

$$t^2 = 36$$

$$\boxed{t = 6 \text{ hours}}$$

score

7. To help focus on studying for calculus you have recently purchased an hourglass as an aide to measure time with a device that does not also beep. When you open the box you are surprised to discover that the top and bottom of the hour glass are differently shaped. One side is a hemisphere with a radius of two inches, and the other side is a pyramid measuring four inches on each side of the base and with a height of three inches. (See the cross section to the right.) At a particular moment you notice that the sand in both the top and bottom of the hourglass have a height of one inch. If the height of the sand in the top is decreasing at a rate of 2/15 inches per minute, at what rate is the height of the sand in the bottom increasing?



Note: the volume in the top is $2\pi H^2 - \frac{1}{3}\pi H^3$; the volume in the bottom is $\frac{16}{27}h^3 - \frac{16}{3}h^2 + 16h$.

Note: (change in top) = − (change in bottom)

$$\left(4\pi H \frac{dH}{dt} - \pi H^2 \frac{dH}{dt}\right) = -\left(\frac{16}{9}h^2\frac{dh}{dt} - \frac{32}{3}h\frac{dh}{dt} + 16\frac{dh}{dt}\right)$$

note $H = h = 1$ and $\frac{dH}{dt} = -\frac{2}{15}$

"−" because sand is dropping

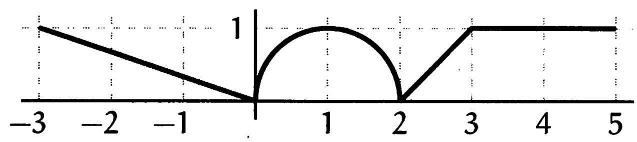
$$(3\pi)\left(\frac{-2}{15}\right) = -\left(\frac{16}{9} - \frac{32}{3} + 16\right)\frac{dh}{dt}$$

$$-\frac{2\pi}{5} = -\frac{64}{9}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2\pi}{5}\cdot\frac{9}{64} = \boxed{\frac{9\pi}{160}\ \frac{\text{inches}}{\text{min}}}$$

score

8. Given $f(x)$ is shown below, and $F(x) = \int_{3-x}^{2x^2} f(t)\,dt$, determine the following.

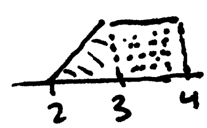


(a) Find $F(1)$.

$F(1) = \int_2^2 f(t)\,dt = \boxed{0}$ (bounds match)

(b) Find $F(-1)$.

$F(-1) = \int_4^2 f(t)\,dt = -\int_2^4 f(t)\,dt = \boxed{-\frac{3}{2}}$



area is $\frac{3}{2}$

(c) Find $F(0)$.

$F(0) = \int_3^0 f(t)\,dt = -\int_0^3 f(t)\,dt = \boxed{-\left(\frac{\pi}{2} + \frac{1}{2}\right)}$

area of $\frac{1}{2}$ unit circle + triangle

(d) Find $F'(-\sqrt{2})$.

$$F'(x) = f(2x^2)\cdot 4x - f(3-x)(-1)$$
$$= 4x\,f(2x^2) + f(3-x)$$

⇐ Fundamental Theorem of Calculus + Chain rule

$$F'(-\sqrt{2}) = -4\sqrt{2}\,\underbrace{f(4)}_{=1} + \underbrace{f(3+\sqrt{2})}_{=1}$$

$$= \boxed{1-4\sqrt{2}}$$

score