1. Find $\dfrac{d}{dx}\left(\dfrac{x^{\sin(x)}\cos(x)}{e^{3x}+5}\right)$.

$$y = \frac{x^{\sin x}\cos x}{e^{3x}+5}$$ ← crazy product + exponent means use logs

$$\ln y = \ln\left(\frac{x^{\sin x}\cos x}{e^{3x}+5}\right)$$

$$= \ln(x^{\sin x}) + \ln(\cos x) - \ln(e^{3x}+5)$$

$$= \sin x \ln(x) + \ln(\cos x) - \ln(e^{3x}+5)$$

take derivative of both sides

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \sin x \cdot \frac{1}{x} + \underbrace{\frac{1}{\cos x}(-\sin x)}_{-\tan x} - \frac{1}{e^{3x}+5}e^{3x}$$

$$= \cos x \ln x + \frac{\sin x}{x} - \tan x - \frac{e^{3x}}{e^{3x}+5}$$

$$\frac{dy}{dx} = \left(\cos x \ln x + \frac{\sin x}{x} - \tan x - \frac{e^{3x}}{e^{3x}+5}\right) y$$

$$\boxed{= \left(\cos x \ln x + \frac{\sin x}{x} - \tan x - \frac{e^{3x}}{e^{3x}+5}\right)\frac{x^{\sin x}\cos x}{e^{3x}+5}}$$

score

2. Find $\int_1^{3\sqrt{3}} \frac{dy}{y^{2/3}(y^{2/3}+1)}$.

$\hookrightarrow$ if $y^{-2/3}$ is part of du then this suggests $u = y^{1/3}$

$$u = y^{1/3}$$

$$du = \frac{1}{3} y^{-2/3} dy \quad \text{or} \quad 3du = \frac{1}{y^{2/3}} dy$$

$$(3\sqrt{3})^{1/3} = \sqrt{3}$$

$$1^{1/3} = 1$$

$$\int_1^{3\sqrt{3}} \frac{dy}{y^{2/3}(y^{2/3}+1)} = \int_1^{3\sqrt{3}} \frac{1}{(y^{1/3})^2+1} \cdot \frac{1}{y^{2/3}} dy$$

$$= \int_1^{\sqrt{3}} \frac{1}{u^2+1} \cdot 3du$$

$$= 3\arctan u \Big|_1^{\sqrt{3}}$$

$$= 3\arctan\sqrt{3} - 3\arctan 1$$

$$= 3 \cdot \frac{\pi}{3} - 3 \cdot \frac{\pi}{4}$$

$$= \pi - \frac{3}{4}\pi$$

$$= \boxed{\frac{1}{4}\pi}$$

score

3. Find $\lim_{t\to 0} \frac{t^2 + 2\ln(\cos t)}{t^4}$. $\leftarrow$ L'Hospital

$$\lim_{t\to 0} \frac{t^2+2\ln(\cos t)}{t^4} \rightarrow \frac{0+2\ln(\cos(0))}{0^4} = \frac{0}{0}$$

$$\overset{L.H.}{=} \lim_{t\to 0} \frac{2t + 2\cdot\frac{1}{\cos t}(-\sin t)}{4t^3} = \lim_{t\to 0} \frac{2t - 2\tan t}{4t^3} \rightarrow \frac{2\cdot 0 - 2\tan(0)}{4\cdot 0^3} = \frac{0}{0}$$

$$\overset{L.H.}{=} \lim_{t\to 0} \frac{2 - 2\sec^2 t}{12t^2} \rightarrow \frac{2-2\sec^2(0)}{12\cdot 0^2} = \frac{0}{0}$$

$$\overset{L.H.}{=} \lim_{t\to 0} \frac{-2\cdot 2\cdot(\sec t)^1\cdot(\sec t\tan t)}{24t} = \lim_{t\to 0} \frac{-4\sec^2 t\tan t}{24t} \rightarrow \frac{4\cdot 1\cdot 0}{24\cdot 0} = \frac{0}{0}$$

$$\overset{L.H.}{=} \lim_{t\to 0} \frac{-4\cdot 2\cdot(\sec t)^1\cdot(\sec t\tan t)\tan t - 4\sec^2 t\sec^2 t}{24}$$

$$= \frac{-4\cdot 2\cdot 1\cdot 0 - 4\cdot 1^4}{24} = \frac{-1}{6}$$

score

4. Consider the function $h(x) = \dfrac{e^x}{4x^2+3}$.

(a) Determine the (maximal) intervals where the function is increasing and, similarly, where the function is decreasing.

$$h'(x) = \frac{(4x^2+3)e^x - e^x \cdot 8x}{(4x^2+3)^2}$$ ← note denominator $\neq 0$

$$= \frac{(4x^2-8x+3)e^x}{(4x^2+3)^2}$$

$4x^2-8x+3=0$

$(2x-3)(2x-1)=0$

$x=3/2,\ x=1/2$

↗ increasing | ↘ dec. | ↗ inc.

$\frac{1}{2}$ $\frac{3}{2}$

$h'(0)=\frac{1}{3}>0$ $h'(1)=\frac{-e}{49}<0$ $h'(2)=\frac{3e^2}{19^2}>0$

$$\boxed{\begin{array}{ll} \text{increasing:} & x \le \frac{1}{2} \text{ and } x \ge 3/2 \\ \text{decreasing:} & \frac{1}{2} \le x \le \frac{3}{2} \end{array}}$$

(b) Find and classify the x-coordinates of the critical point(s) of $h(x)$.

critical points are $\frac{1}{2}, \frac{3}{2}$ using first derivative test (from above)

$$\boxed{\begin{array}{ll} x=\frac{1}{2} & \text{local max} \\ x=\frac{3}{2} & \text{local min} \end{array}}$$

score

5. Find and verify the value of $a > 0$ so that the *average value* of $f(x) = 30x - 4x^3$ on the interval $0 \leqslant x \leqslant a$ is maximal.

average value for $0 \le x \le a$

$$= \frac{1}{a-0} \int_0^a (30x - 4x^3)\,dx$$

$$= \frac{1}{a} (15x^2 - x^4)\Big|_0^a$$

$$= \frac{1}{a}(15a^2 - a^4)$$

$$= 15a - a^3$$

Let $f(a) = 15a - a^3$ $\Leftarrow$ what we are optimizing

$f'(a) = 15 - 3a^2$

$15 - 3a^2 = 0$ if $3a^2 = 15$ if $a^2 = 5$ if $a = \sqrt{5}$

$$\boxed{a = \sqrt{5}}$$

Verification: $f''(a) = -6a$

$f''(\sqrt{5}) = -6 \cdot \sqrt{5} < 0$ so $a = \sqrt{5}$ is a <u>max</u>

score

6. Rewrite the following as a *single* integral, i.e., of the form $\int_a^b f(u)\,du$:

$$\int_0^{\ln(3)} e^x f(e^x)\,dx + \int_3^6 \sin(2\pi x) f(\sin(\pi x))\,dx + \int_6^{10} \tfrac{1}{2} f(8 - \tfrac{1}{2}x)\,dx.$$

$$\int_0^{\ln(3)} e^x f(e^x)\,dx = \int_1^3 f(u)\,du$$

$u = e^x \qquad e^{\ln(3)} = 3$

$du = e^x dx \qquad e^0 = 1$

bounds agree

$$\int_3^6 \sin(2\pi x) f(\sin(\pi x))\,dx = \int_3^6 2\sin(\pi x)\cos(\pi x) f(\sin(\pi x))\,dx = \frac{2}{\pi}\int_0^0 u f(u)\,du = 0$$

$u = \sin(\pi x) \qquad \sin(3\pi) = 0$

$du = \pi\cos(\pi x)dx \qquad \sin(6\pi) = 0$

$$\int_6^{10} \tfrac{1}{2} f(8 - \tfrac{1}{2}x)\,dx = \int_5^3 f(u)(-du) = \int_3^5 f(u)\,du$$

$u = 8 - \frac{1}{2}x \qquad 8 - \frac{1}{2}\cdot 10 = 3$

$du = -\frac{1}{2}dx \qquad 8 - \frac{1}{2}\cdot 6 = 5$

$$\int_1^3 f(u)\,du + 0 + \int_3^5 f(u)\,du = \boxed{\int_1^5 f(u)\,du}$$

score

$$f'(x) = 3x^2 + 7$$

7. In this problem use Newton's method to approximate a root to $f(x) = x^3 + 7x - 3$.
(a) Give the recurrence for x_{n+1} as a function of x_n. (The answer should not use "f".)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 + 7x_n - 3}{3x_n^2 + 7}}$$

(b) Starting with $x_0 = 2$, give the *exact* value of x_3.

$$x_0 = 2$$

$$x_1 = 2 - \frac{8 + 14 - 3}{12 + 7} = 2 - \frac{19}{19} = 2 - 1 = 1$$

$$x_2 = 1 - \frac{1 + 7 - 3}{3 + 7} = 1 - \frac{5}{10} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$x_3 = \frac{1}{2} - \frac{\frac{1}{8} + \frac{7}{2} - 3}{\frac{3}{4} + 7} = \frac{1}{2} - \frac{1 + 28 - 24}{6 + 56}$$

$$= \frac{1}{2} - \frac{5}{62}$$

$$= \frac{31}{62} - \frac{5}{62}$$

$$= \frac{26}{62} = \boxed{\frac{13}{31}}$$

score

8. Given that $y' = (y+1)e^x$ and that $y(0) = e - 1$, find an expression for $y(x)$.

$$\frac{dy}{dx} = (y+1)e^x$$

$$\frac{dy}{y+1} = e^x dx$$

$$\int \frac{dy}{y+1} = \int e^x dx$$

$$\ln(y+1) = e^x + C$$

← initial conditions

$$\underbrace{\ln(e-1+1)}_{\underbrace{\ln(e)}_{1}} = \underbrace{e^0}_{1} + C$$

$1 = 1 + C$ so $C = 0$

$$\ln(y+1) = e^x$$

$$y+1 = e^{e^x}$$

$$\boxed{y = e^{e^x} - 1}$$

score