Algebra

Algebra is the foundation of calculus. The basic idea behind algebra is rewriting equations and simplifying expressions; this includes such things as factoring, FOILing (i.e., $(a+b)(c+d) = ac+ad+bc+bd$), adding fractions (remember to get a common denominator, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$) and multiplying by the conjugate (the conjugate of $a + b$ is $a - b$ and multiplying gives $a^2 - b^2$). Below are some basic facts that frequently come up.

$$\begin{aligned} (a+b)(a-b) &= a^2 - b^2 \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2 \end{aligned}$$

Another useful fact is the quadratic formula (also known as the quadratic equation)

$$ax^2 + bx + c = 0 \;\Rightarrow\; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Rules for exponents and logarithms are useful:

$$x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad (x^a)^b = x^{ab},$$
$$\ln(ab) = \ln a + \ln b, \quad \ln\left(\frac{a}{b}\right) = \ln a - \ln b,$$
$$\text{and} \quad \ln(a^b) = b \ln a.$$

Of course it is also useful to remember what we **cannot** do. For example, "$\sqrt{x^2 + a^2} = x + a$" is **not** true, even when taking a test!

Often associated with algebra are functions and graphing. A function is a rule which takes an input and associates a unique output, i.e., $y = f(x)$. The domain of a function are numbers that we can put into the function and get something out, while the range are the possible values that we can get out. Limitations on the domain arise from dividing by zero, taking square roots of negative numbers, or log of a nonpositive number. Determining the range is hard, but fortunately there is calculus (woohoo!). Functions can be combined through addition, multiplication, division, composition, etc.

We work with points in the plane by describing them relative to how far away from the origin we have moved in different directions, i.e., (x, y). The distance between the two points (x_0, y_0) and (x_1, y_1) in the plane is

$$\text{distance} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

We have lines ($y = mx + b$ where m is the slope (found by rise/run) and b the y-intercept), circles ($(x - x_0)^2 + (y - y_0)^2 = r^2$ where (x_0, y_0) is the center of the circle and r the radius), parabolas ($y = ax^2 + bx + c$), ellipses ($ax^2 + by^2 = 1$) and hyperbolas ($ax^2 - by^2 = 1$).

The area of a rectangle is (length)×(width), the area of a circle is πr^2 (where r is the radius) and the area of a triangle is $\frac{1}{2}$(base)×(height). The volume of a box is (length)×(width)×(height) and the volume of a sphere is $\frac{4}{3}\pi r^3$ (again r is the radius).

Trigonometry

Trigonometry is based off the two ideas that (i) triangles are rigid and (ii) when we scale a triangle the ratios of the sides do not change. Using these we can associate values with the angles of triangles which can be used to solve various problems related to triangles. The basis of many important facts about triangles comes from the Pythagorean Theorem which says that for a right triangle with sides a, b, c (c being the hypotenuse) then $a^2 + b^2 = c^2$. What this means is that often when dealing with triangles we look for right triangles. Let θ be an angle of a right triangle, "opp" the length of the side opposite θ, "adj" the length of the side adjacent to θ and "hyp" the length of the hypotenuse; then we have

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}},$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}.$$

(We either measure angles in degrees (360° is a full revolution) or radians (2π is a full revolution). In calculus we almost always will use radians.) Note that the functions sin and cos are periodic and also are always bounded between -1 and 1 (← a very useful fact).

From the definitions have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{and} \qquad \sec\theta = \frac{1}{\cos\theta}.$$

And by using the Pythagorean Theorem we can conclude

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{and} \qquad \tan^2\theta + 1 = \sec^2\theta.$$

Other rules which frequently come up are the double angle formulas

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta, \end{aligned}$$

this can be used to reduce powers of sine and cosine

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \quad \text{and} \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

We also have the even/odd'er identities

$$\sin(-\theta) = -\sin\theta, \quad \text{and} \quad \cos(-\theta) = \cos\theta.$$

There are of course many more formulas and identities that arise in trigonometry, but for our purposes these will usually suffice (we will make use, for example, of the law of cosines but in general these are the most important identities).

Quiz 1 problem bank

1. Find the domain for the function $f(x) = \dfrac{\sqrt{9-x^2}}{\ln(2-e^x)}$.

2. Find $f^{-1}(x)$ given $f(x) = \frac{1}{2}(e^x - e^{-x})$.

3. Find the slope and y-intercept for the line $y = 5 + \ln(7e^{5x})$.

4. Given $f(x) = x + 3$ and $g(x) = x^2 + 4$, then $h(x) = g(f(x)) - f(g(x))$ is a line. Find the slope and y-intercept for $h(x)$.

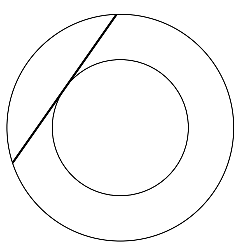
5. Sketch the curve $y = 2 - x - |x|$ for $-2 \leqslant x \leqslant 2$.

6. Find all solutions (x, y) to the following:
$$\begin{aligned} xy - x + 2y &= 2 \\ x^2 + 5x + 4y &= 0 \end{aligned}$$

7. For $-1 \leqslant x \leqslant 1$, rewrite $\cos\left(2\arccos\sqrt{\dfrac{x+1}{2}}\right)$ as an algebraic expression.

8. For $x > 1$, rewrite $\cos\left(\arctan\left(\dfrac{2x}{x^2-1}\right)\right)$ as an algebraic expression.

9. Given that the circles in the following diagram have the same center and that the length of the line segment is 20, determine the area *between* the circles.



10. Given for the diagram below that $AB = AC = CD$ and $AD = BD$, determine the angle α, give the answer in radians.