Newton's method

Given a function $f(x)$ we might be interested in finding the roots of the function. These are values c where $f(c) = 0$. For some functions this can easily be done (e.g., the quadratic formula), but in general this is a hard problem. However, if we know the approximate value of the root then we can work to get an ever better approximation (i.e., we can sometimes get good estimates for our roots).

One such approach is Newton's method which makes the following observation. Suppose that c is a guess for our root, then it is reasonable to use the tangent line at c as a stand-in for the function and find when the tangent line has a zero to find a better approximation. Note that the tangent line at c is $y = f(c) + f'(c)(x - c)$ and when we solve this for when the line is zero we get our new guess:

$$x = c - \frac{f(c)}{f'(c)}$$

In general we think about iterating this process so that we have a series of guesses, $x_0, x_1, \ldots$ where each one is (hopefully) better and

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}.$$

If we are looking for a root of $f(x) = x^2 - 2$ (hint: this is $\sqrt{2}$), then the recursion becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}x_n + \frac{1}{x_n}.$$

Letting $x_0 = 1$ this gives:

$$\begin{aligned} x_0 &= 1 &&\approx 1.0000 \\ x_1 &= \tfrac{3}{2} &&\approx 1.5000 \\ x_2 &= \tfrac{17}{12} &&\approx 1.4166 \\ x_3 &= \tfrac{577}{408} &&\approx 1.4142 \end{aligned}$$

Note that this works best when we start near a root (though it might not get near the root you are looking for). Moreover it is possible that the process will get into a loop or even blow up!

Antiderivatives

An antiderivative of $f(x)$ is a function $F(x)$ so that $F'(x) = f(x)$. As an example for the function $f(x) = 2x$ an antiderivative is $F(x) = x^2$, but another antiderivative is $F(x) = x^2 + 1$ and in general any function of the form $F(x) = x^2 + C$ for C a constant is an antiderivative of $f(x)$, but these are the *only* antiderivatives of $f(x) = 2x$. In general, if $F(x)$ is an antiderivative of $f(x)$ then all antiderivatives are of the form $F(x) + C$. Or in other words two antiderivatives only differ by a constant.

Notationally we will let $\int f(x)\,dx$ denote the antiderivative of $f(x)$ or the "indefinite integral" of $f(x)$. So we have that

$$\int f(x)\,dx = F(x) + C,$$

where $F(x)$ is any antiderivative of $f(x)$.

Not surprisingly, rules for derivatives become rules for antiderivatives. So all the following rules are found by taking rules for derivatives and rewriting them in terms of rules for antiderivatives.

$$\begin{aligned} \int \big(f(x) + g(x)\big)\,dx &= \int f(x)\,dx + \int g(x)\,dx \\ \int (k{\cdot}f(x))\,dx &= k\int f(x)\,dx \\ \int x^a\,dx &= \frac{1}{a+1}x^{a+1} + C \quad (a \neq -1) \\ \int \frac{1}{x}\,dx &= \ln x + C \\ \int e^x\,dx &= e^x + C \\ \int \sin x\,dx &= -\cos x + C \\ \int \cos x\,dx &= \sin x + C \\ \int \sec^2 x\,dx &= \tan x + C \\ \int \sec x \tan x\,dx &= \sec x + C \\ \int \frac{1}{1+x^2}\,dx &= \arctan x + C \end{aligned}$$

Note that the sine and cosine now have "swapped signs" compared to the corresponding rules for their derivatives.

For any given function there are many antiderivatives (since again we can add an arbitrary constant); but we might be interested in finding one specific antiderivative. This is possible, for example, if we know about the value of the function at a point since this lets us determine which C is the correct one. For example suppose we know that $y' = 4x^7 - 2\sin x$ and that $y(0) = 4$ and we want to find y. First we note that y will be an antiderivative of y', i.e.,

$$\begin{aligned} y = \int (4x^7 - 2\sin x)\,dx &= 4\int x^7\,dx - 2\int \sin x\,dx \\ &= \frac{4}{8}x^8 + 2\cos x + C = \frac{1}{2}x^8 + 2\cos x + C. \end{aligned}$$

But we also know that $y(0) = 4$ and so we can use this to find C, for example we have

$$4 = y(0) = 0 + 2 + C \quad\text{or}\quad C = 2.$$

So the desired function is $y = \frac{1}{2}x^8 + 2\cos x + 2$. This is a simple example of what is known as an "initial value problem", i.e., we know something about how the derivative is behaving and an initial value of the function and we want to determine the function.

It turns out that integration can be much trickier than differentiation. As a general rule if you have a doubt whether or not what you are doing is correct then don't! (A lot of "intuitive" things do not work.) In fact we will spend a *significant* chunk of next semester finding ways of rewriting expressions so that it looks like one of the ones that we have listed above.

Quiz 10 problem bank

1. For $f(x) = x^3 - 3x + 1$, find an expression for the recurrence for Newton's method (i.e., x_{n+1} =(stuff with x_n)). Given $x_0 = 2$, find x_1.

2. For $g(x) = x^3 - 7$ use Newton's method starting with $x_0 = 2$ to find x_1 and x_2, give your answer to four decimal places.

3. Given that $x_{n+1} = \dfrac{4x_n}{5} + \dfrac{4}{x_n^4}$ with $x_0 = 137$, determine the *exact value* that the numbers x_n are approaching as n gets large.

4. Let $f(\theta) = \cos(\theta) + 2\cos(\theta)\sin(\theta) - \sin(\theta)$, let $g(\theta) = \frac{1}{2}(\cos(\theta) + \sin(\theta) + 1)^2$ and let $h(\theta) = \cos(\theta) + \sin(\theta) + \frac{1}{2}\sin(2\theta)$. Which of $g(\theta)$ or $h(\theta)$ an antiderivative of $f(\theta)$? Explain.

5. Find $\displaystyle\int \frac{e^{3x} + 1}{e^x + 1}\,dx$.

6. Find $\displaystyle\int \frac{x^3 + 3\sqrt{x} + 5}{x}\,dx$.

7. Find $\displaystyle\int \frac{1}{1 + \sin\theta}\,d\theta$.

8. Find $\displaystyle\int |t|\,dt$.

9. Given $f'(x) = \dfrac{x^2}{1 + x^2}$ and $f(1) = 4$, find $f(x)$.

10. Suppose that $y''(t) = 2 - \sin(\pi t)$ and that $y(0) = -2$ and $y(2) = 5$. What is $y'(1)$?