Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the ideas of differentiation with our new idea of integration. There are two parts to the Fundamental Theorem of Calculus.

> (Part I) If f is continuous on $[a,b]$ then
> $$\int_a^b f(u)\,du = F(b) - F(a),$$
> where $F(x)$ is *any* anti-derivative of $f(x)$.

In particular if we want to evaluate a definite integral we can now do it in two steps. First, find an anti-derivative of the function. Second, evaluate this new function at the endpoints and take the difference. This reduces the problem of integration to that of finding an anti-derivative.

Of course finding anti-derivatives in general are not easy! Our main technique is to work on rewriting the function using algebraic manipulation, trigonometric identities, or substitution (see below) so that we can reduce the anti-derivative to something that we easily recognize.

The other part of the Fundamental Theorem of Calculus says that integration leads to anti-derivatices.

> (Part II) If f is continuous on $[a,b]$ and
> $$F(x) = \int_a^x f(u)\,du \ \text{ then } \ F'(x) = f(x).$$

By combining this part of the Fundamental Theorem of Calculus, the chain rule and properties of integrals we have the following rule:

$$\frac{d}{dx}\left(\int_{h(x)}^{g(x)} f(u)\,du\right) = f\big(g(x)\big)g'(x) - f\big(h(x)\big)h'(x).$$

We see from part II of the Fundamental theorem of calculus that the function $F(x)$ is an anti-derivative of $f(x)$. So we will let $\int f(x)\,dx$ (called the indefinite integral) denote the anti-derivative of $f(x)$. In general we have that

$$\int f(x)\,dx = C + \int_a^x f(x)\,dx,$$

where C is a constant (this constant will play an important role later, it is important not to forget it).

Substitution rule

Rules for derivatives become rules for integration. One of the most important rules for derivatives is the chain rule which states

$$\frac{d}{dx}\big(f\big(g(x)\big)\big) = f'\big(g(x)\big)g'(x).$$

By taking the anti-derivative of each side we can conclude

$$\int f'\big(\underbrace{g(x)}_{=u}\big)\underbrace{g'(x)\,dx}_{=du} = \int f'(u)\,du = f(u) + C$$
$$= f\big(g(x)\big) + C.$$

This is used in *many* problems involving integration because it can help rewrite the integral in a simpler form. So after the substitution we might see how to proceed and then we can solve the integral and at the end *resubstitute* back to get our answer in terms of x. The indication that we should use use substitution is to look for a function inside of a function.

With every method we have in working with solving integrals the goal is always to make it simpler. In some sense the art of integration is the art of cumulative simplification. It is possible that several substitutions might be needed. This is fine as long as you keep track of everything.

Remember that when we are substituting that we need to substitute for every occurrence of "x", i.e., we also need to make sure we substitute for the dx term. On some integrals, in order for us to do this we might need to solve for x in terms of u. For example when making the substitution $u = \sqrt{x}$ then $du = \frac{1}{2}x^{-1/2}\,dx$ or $dx = 2\sqrt{x}\,du = 2u\,du$ so that the appropriate substitution in this case is to replace the dx term by $2u\,du$. (Note it is easy to add and divide by constants to get what we need.)

If we are dealing with a definite integral we can do one of two things. First, we do the indefinite integral, solve it to the end to get an antiderivative and then use the fundamental theorem of calculus to evaluate and get our answer. Alternatively, we can change the bounds as we make our substitution (the principle is again that we are replacing *every* occurrence of x, and the original bounds were in terms of x, i.e., $\int_a^b(\text{stuff})\,dx$ indicates we go from $x = a$ to $x = b$). So we have

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \ \text{where } u = g(x).$$

Applications of integration

Cumulative change: There is a connection between integration and derivatives and we can use this to answer questions about given how fast something is changing, what is the total. (These types of questions are easy to identify since they will involve only one rate (where related rates involves more than one) and will ask for a total.) By the fundamental theorem of calculus we have

$$\int_a^b f'(t)\,dt = f(b) - f(a) \ \text{ or } \ f(b) = f(a) + \int_a^b f'(t)\,dt.$$

The intuition is that $f'(t)\,dt$ measures the (instantaneous) amount of change at time t and then the "$\int$" adds them all up to find the total amount of change.

If $v(t) = s'(t)$ is velocity then

$$\int_a^b v(t)\,dt = s(b) - s(a) = \text{displacement},$$

and

$$\int_a^b |v(t)|\,dt = \text{total distance},$$

Area: We can use integration to find the area between curves. If $g(x) \leqslant f(x)$ on the interval $[a, b]$ then the area between these curves in the interval is

$$\text{Area} = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

If the curves cross then find the intersection point(s) by setting $f(x) = g(x)$ and solving for x (also done when no bounds are given, or when there are several curves that define the region). Once we have the intersection point(s) we split this into several pieces and work on each piece separately (this is to avoid the problem of "signed" areas and to simplify the integrals to manageable functions).

We can also integrate with respect to y, the basic idea being to take horizontal slices. In this case if we have $x = f(y)$ and $x = g(y)$ with $g(y) \leqslant f(y)$ on the interval $[a, b]$. Then we have

$$\text{Area} = \int_a^b \big(f(y) - g(y)\big)\,dy.$$

Average value: We already saw that the way to compute the average value of f in the interval $[a, b]$ is by

$$\text{average} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This value is such that the rectangle with height f_{avg} and width $(b - a)$ has the same area as $\int_a^b f(x)\,dx$.

The *Mean Value Theorem for Integrals* states that if f is continuous on $[a, b]$ then there is some $c \in [a, b]$ so that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(This is actually the same as the Mean Value Theorem for derivatives, just worded differently.)

Quiz 12 problem bank

1. Given $F(x) = \displaystyle\int_{2x^2}^{x^3} \sqrt{2 + \sqrt[3]{t}}\,dt$, find the tangent line to $F(x)$ at $x = 2$.
2. For $x \geqslant 0$, find $F(x) = \displaystyle\int_0^x 6|t^2 - t|\,dt$. (Hint: find $F(x)$ as a piecewise function according to how we can break up the function inside the integral.)
3. Find $\displaystyle\int \frac{\sin(\sqrt{x})\sin(2\sqrt{x})}{\sqrt{x}}\,dx$.
4. Find $\displaystyle\int_0^2 \sqrt{t^4 + 9}\,dt + \int_3^5 \sqrt[4]{t^2 - 9}\,dt$.
5. Find $\displaystyle\int \sqrt{1 + \sqrt{x}}\,dx$.
6. Let $h(x) = \displaystyle\int_{3x^2-2}^{2x^2+x} \frac{1}{2 + \sin t}\,dt$. Determine $h'(x)$.
7. Reduce the following to a single integral of the form $A\displaystyle\int_B^C f(x)\,dx$ for some constants A, B, C.
$$\int_0^5 f(x)\,dx - \int_3^3 f(x^2)\,dx + \int_0^1 3f(3x)\,dx - \int_0^4 f\big(\tfrac{1}{2}x\big)\,dx + \int_5^3 f(x)\,dx.$$
8. Find $\displaystyle\int \frac{3x}{\sqrt{x^2+1} + x}\,dx$.
9. Given that $\displaystyle\int_3^x g(t)\,dt = \sqrt[3]{x^2 - 1} + Cx$, find C and $g(x)$.
10. Find $\displaystyle\int_0^1 2x\sqrt{1 - x^4}\,dx$.