L'Hospital's formula

At the beginning of the class we developed limits to help find derivatives which involved expressions which went to $\frac{0}{0}$. Now we can come full circle and use derivatives to help us find limits that are of the form $\frac{0}{0}$. We have the following.

> (L'Hospital's Formula) If $f(x)$ and $g(x)$ are differentiable around $x = a$ and if as $x \to a$ we have $\frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\to \frac{\infty}{\infty}$, then
>
> $$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}.$$

This follows by noting that we can manipulate the limit to a derivative. As an example we have the following:

$$\lim_{\theta\to 0}\frac{\sin(5\theta)}{e^{2\theta}-1} = \lim_{\theta\to 0}\frac{5\cos(5\theta)}{2e^{2\theta}} = \frac{5}{2}.$$

If needed we might need to apply this several times, we keep going until we can make a decision about the limit. We can use this to establish some basic facts. For example we have the following, suppose that the function $f(x)$ has a second derivative, then

$$\begin{aligned}\lim_{h\to 0}&\frac{f(x+h)-2f(x)+f(x-h)}{h^2}\\ &= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\ &= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2} = f''(x).\end{aligned}$$

(Note in this example the variable we took the derivative with respect to was h as that is what is in the limit.)

The expressions $\frac{0}{0}$ and $\frac{\infty}{\infty}$ are not the only times we cannot decide what is happening, other indeterminate forms include $\frac{\infty}{\infty}$, $0\cdot\infty$, 1^∞, $\infty-\infty$, and so on. The typical approach is to find a way to rewrite these expressions so that it approaches either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. One particular interesting variation is when our limit involves an expression where both the base and exponent are changing. In this case it is helpful to observe the following two ideas. First the logarithm function is continuous. Second if $f(x)$ is continuous then

$$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big).$$

So for example suppose we want to find

$$\lim_{x\to\infty}\Big(1+\frac{t}{x}\Big)^x$$

which is approaching 1^∞ we let y be our answer and then note

$$\begin{aligned}\ln(y) = \ln\bigg(\lim_{x\to\infty}\Big(1+\frac{t}{x}\Big)^x\bigg) &= \lim_{x\to\infty}\ln\Big(1+\frac{t}{x}\Big)^x\\ &= \lim_{x\to\infty} x\ln\Big(1+\frac{t}{x}\Big) = \lim_{x\to\infty}\frac{\ln\big(1+\frac{t}{x}\big)}{\frac{1}{x}}\end{aligned}$$

Now we can apply L'Hospital and this becomes

$$\ln(y) = \lim_{x\to\infty}\frac{\frac{-t/x^2}{1+t/x}}{-1/x^2} = \lim_{x\to\infty}\frac{t}{1+t/x} = t.$$

Now we remember that we wanted y, so to finish we have that $\ln(y) = t$ becomes $y = e^t$.

Separable differential equations

An important part of mathematics is *differential equations* which relates how a function is changing (i.e., its derivative) with the current value of the function and/or the input of the function. (We have previously done a very special case of this when the derivative is only in terms of the input and then recovering the original function.)

The problem of solving differential equations is very hard and is still being worked on. There are a few special cases where we can make progress and we will do one here (don't worry we will get to spend a whole semester on these later). In particular we consider *separable* differential equations which are of the form

$$y' = \frac{f(x)}{g(y)}.$$

The key observation is that we can rewrite this as

$$g(y)y' = f(x).$$

We now think of both sides as functions of x (wait for it) and since both sides are equal there antiderivatives are also equal. The antiderivative of $f(x)$ is a typical antiderivative problem. For $g(y)y'$ we note

$$\int g(y)y'\,dx = \int g(u)\,du$$

by making the substitution $u = y$ and $du = y'\,dx$, so this is also an antiderivative problem (there it is!). So taking antiderivatives of both sides they agree up to a constant and then we use additional information (i.e., some initial value) to determine the constant. Note, we can sometimes solve for y, but it is usually best to solve for the constant straight away (i.e., once we have taken antiderivatives).

This process is best seen with an example. So suppose we have that $y' = -\frac{1}{10}y$ and $y(0) = 20$ where y is a function of t (this is typical of things such as radioactive decay where things decrease in proportion to the amount of material). We separate by putting all of the variables of one type on one side and all the other variables on the other

$$\frac{y'}{y} = -\frac{1}{10} \quad \text{integrating gives} \quad \ln(y) = -\frac{1}{10}t + C.$$

(Note we only need a "+C" on one side, i.e., we can combine several constants to form a single uber-constant.) Solving for y then we get

$$y = e^{-t/10+C} = e^{-t/10}e^{C} = De^{t/10}.$$

Finally to get the constant we use the initial condition that at time 0 we have $y = 20$ so this gives $D = 20$,

$$y = 20e^{-t/10}.$$

Of course there are *many* other possibilities. Suppose that $y' = \frac{x}{2y+4}$ and $y(\sqrt{7}) = 0$. First we separate:

$$(2y+4)y' = x$$

Then we integrate:

$$y^2 + 4y = \frac{1}{2}x^2 + C$$

Then we solve for C and then y; or we solve for y and then C. In this case, let's solve for C first. We have

$$0 = \frac{1}{2}\sqrt{7}^2 + C \quad \text{giving} \quad C = -\frac{7}{2}.$$

Updating we now have

$$y^2 + 4y = \frac{1}{2}x^2 - \frac{7}{2}.$$

Now to solve for y we note it would be helpful to complete the square on the left by adding 4 giving

$$y^2 + 4y + 4 = \frac{1}{2}x^2 - \frac{7}{2} + 4 \quad \text{or} \quad (y+2)^2 = \frac{1}{2}(x^2+1).$$

Finally we solve by taking the square root and subtracting 2 giving

$$y = -2 + \sqrt{\frac{1}{2}(x^2+1)}.$$

(Note there were actually two possibilities when we took the square root, i.e., $\pm$; both should be checked, if we had gone with the "$-$" then we could not still satisfy the initial conditions, i.e., $y = -2 - \sqrt{\frac{1}{2}(x^2+1)}$ has $y(\sqrt{7}) = -4 \neq 0$.)

Quiz 13 problem bank

1. Find $\displaystyle\lim_{\theta\to 0} \frac{3\sin(\theta) - \sin(3\theta)}{\theta^3}$.
2. Find $\displaystyle\lim_{x\to 0} \frac{\cos(3x) - 1 - x^2}{e^x + e^{-x} - 2}$.
3. Find $\displaystyle\lim_{t\to\infty} \frac{t^{137}}{e^t}$.
4. Find $\displaystyle\lim_{x\to\infty} \big(1 - e^{-x}\big)^x$.
5. Find $\displaystyle\lim_{x\to 0} \big(\cos x\big)^{1/x^2}$.
6. Now, the Star-Belly Sneetches had bellies with stars. The Plain-Belly Sneetches had none upon thars. Then one day Sylvester McMonkey McBean came to town with his wondrously wonderful machine, "Just one pass through, hop on board, and you will have a star for sure." The Sneetches listened and the Sneetches thought and those who wanted a star-belly stepped up and bought. Sylvester kept track of the proportion of Sneetches with stars (P) and noticed with time (t) in months that $P' = \frac{1}{3}t(1-P)^2$. His business was quick, he did not want to delay, and so he recalled on his very first day that $Q = 1/4$, to make a quick buck and then leave this place he decided to leave when $Q = 3/4$. How many months then will it take until Sylvester McMonkey McBean leaves this place?
7. It has recently been revealed that the reason that twinkies have such an incredible shelf life is due to a rare molecule known as Twinkonium (T). Each twinkie contains five grams of Twinkonium when first produced, unfortunately the Twinkonium then starts to decay and turn into ordinary sugar, through extensive research it has been determined that the rate of decay satisfies $T' = -\frac{1}{10}T^2$, where t is time measured in months. If a twinkie is still good when it has at least one gram of Twinkonium, how many months from when it was first produced will the twinkie go bad?
8. Find y given $y' = 2x(y^2+1)$ and $y(2) = 0$
9. Find y given $y' = \dfrac{x}{e^{2y} + e^{y}}$ and $y(\sqrt{3}) = 0$.
10. [Mystery problem]