Rate of change

We are interested in finding the rate of change of a function. In particular, given a function $y = f(x)$ we are interested in finding how fast y is changing with respect to x at some fixed time $x = a$. The main way we will do this is to combine two observations.

1. Given a line $y = mx + b$ the slope is m and is found by
$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$
In other words, for a line the rate of change of y with respect to x is the slope.

2. For a typical function we will encounter, when we look at the function "near" $x = a$ it will look like a line, namely the tangent line. (The *tangent line* is the line which "touches without crossing" the curve.)

So to find the rate of change we will find the slope of the tangent line. But, to calculate the slope we need two points and the tangent line only gives one. So first we work with the simpler case of the *secant line* (a secant line crosses the curve at two values of x).

The secant line which intersects the curve $y = f(x)$ at $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ has slope
$$\frac{f(b) - f(a)}{b - a},$$
or if we let $b = a + h$ this can be written as
$$\frac{f(a + h) - f(a)}{h}.$$
This slope gives the *average rate of change* of $y = f(x)$ from $x = a$ to $x = b$, i.e., the constant rate that f would have to change at to go from $\big(a, f(a)\big)$ to $\big(b, f(b)\big)$.

The slope of the secant line will approximate the slope of the tangent line and the approximation will get better and better as b gets closer to a (or equivalently as h goes to 0). The problem is that if $a = b$ or if $h = 0$ then these slopes are $0/0$ which are undefined, so we need some way to handle this.

Limits

The way we handle this is to use *limits*. Intuitively limits tell us what *should* happen based on what is happening *nearby*. So for example
$$\lim_{x \to c} g(x) = L,$$
which we read "the limit as x goes to c of $g(x)$ is L", means that as x gets close to c the function $g(x)$ is getting close to L (and staying close!). It is possible that the limit does not exist. For example,
$$\lim_{x \to 0} \sin\Big(\frac{1}{x}\Big) = \text{ Does not exist.}$$
To see this we note that the function $\sin(1/x)$ will do "infinitely" many oscillations between 1 and -1 around $x = 0$ and so it does not approach a single fixed L.

One way to guess a limit is to plug in values of x closer and closer to c and see if it is approaching some certain value; we can also try plotting a picture of $g(x)$ near $x = c$ and seeing how the function is behaving. Both of these methods have short comings, in particular they are hard to do without a calculator and can sometimes be deceiving. So we want to have some methods to deal with these limits. One method is to build up a collection of rules that we can use. For example we have the following two rules
$$\lim_{x \to c} k = k \quad \text{and} \quad \lim_{x \to c} x = c.$$
(The first follows by noting that k is always close to k, and the second says "as x gets close to c then x gets close to c.) On the other hand we have that if
$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$
and L, M are finite then we have the following rules, which essentially say that limits do what we think they should do.

1. $\lim_{x \to c} \big(f(x) + g(x)\big) = \big(\lim_{x \to c} f(x)\big) + \big(\lim_{x \to c} g(x)\big)$.
2. $\lim_{x \to c} \big(kf(x)\big) = k\big(\lim_{x \to c} f(x)\big)$
3. $\lim_{x \to c} \big(f(x)g(x)\big) = \big(\lim_{x \to c} f(x)\big)\big(\lim_{x \to c} g(x)\big)$.
4. $\lim_{x \to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\big(\lim_{x \to c} f(x)\big)}{\big(\lim_{x \to c} g(x)\big)}$ (when $M \neq 0$).
5. $\lim_{x \to c} \big(f(x)\big)^q = L^q$ (where $q > 0$ is rational).

From these rules we have that the limits of polynomials are found be evaluating the polynomial at the limit point. Similarly for ratio of polynomials if the denominator is not 0.

Squeeze Theorem

One way to find a limit of a function that we do not understand is to put it between two functions that we do understand that come together. In particular if we have
$$\ell(x) \leqslant f(x) \leqslant u(x)$$
for x near c and
$$\lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = K \quad \text{then} \quad \lim_{x \to c} f(x) = K.$$

Related to this is the observation that if $f(x) \leqslant g(x)$ for x near c and the limits for both f and g exist near c, then $\lim_{x \to c} f(x) \leqslant \lim_{x \to c} g(x)$.

Algebraic manipulation of limits

When our limit is going to $0/0$ (or possibly ∞/∞, $\infty - \infty$, $0{\cdot}\infty$, etc.) then we have an ambiguous number since $0/0$ is undefined. Most of the time when we encounter this we will try to manipulate what we are taking the limit of, the goal being to "cancel the 0s" (i.e., rewrite it in such a way that we can cancel a common term from top and bottom so that what remains does not go to $0/0$). There are three main techniques we can use.

1. **Rewriting**. This is usually done when we have a polynomial and we can either expand the polynomials out or factor (or sometimes both).
2. **Multiplying by the conjugate**. The conjugate of an expression $a - b$ is $a + b$. So if we multiply both top and bottom by the conjugate of $a - b$ and then multiply out we get $a^2 - b^2$ (this can be helpful for instance in getting rid of square roots). The reason we have to multiply both top and bottom is so that we do not change the limit, i.e., multiplying by 1 does not change the value of the limit.
3. **Using identities**. This is most commonly done with limits involving trigonometry in which case there are often many identities which we can use to rewrite (and hopefully cancel!) the terms.

Rigorous approach to limits

Intuitively, $\lim\limits_{x\to c} f(x) = L$ is saying that as x gets close to c then $f(x)$ gets close to L. We can make this rigorous (and thus ensure that what we are doing will actually produce meaningful answers). The key is to understand "close". In particular, by close we are talking distance so we want to say that the distances are small. The distance between numbers is found by absolute values, so we have "x gets close to c" becomes $|x - c| < \delta$ (where δ is some number to measure how close), and "$f(x)$ gets close to L" becomes $|f(x) - L| < \varepsilon$ (where ε is again some number to measure how close).

The key observation to make is that we want to ensure that no matter how close we want $f(x)$ to be to L we can guarantee this by ensuring that x is close to c. Hence we have the following formal definition.

> We have $\lim\limits_{x\to c} f(x) = L$ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

This leads to the classic game of you give me an ε and I will find a δ. From this we can rigorously justify all of the intuitive rules we mentioned earlier. Note one useful method in doing so is the "add 0" method.

Quiz 2 problem bank

1. Find the average rate of change of the function $f(x) = x^3 - 4x^2 + 5x - 3$ over the interval $[0, 2]$.
2. Given that the average rate of change for $y = f(x)$ over the interval $[0, 3]$ is -1, the average rate of change over the interval $[2, 3]$ is 5, and the average rate of change over the interval $[2, 6]$ is 3, determine the average rate of change over the interval $[0, 6]$.
3. Find the average rate of change of $y = x^2$ from $x = a$ to $x = b$. If possible, use algebra to simplify the expression.
4. Determine the unique $c < 0$ so that for $f(x) = x^3 - 2x$ the average rate of change between c and 1 equals the average rate of change between 1 and 2.
5. Find $\displaystyle\lim_{x\to 2} \frac{\sqrt{x+2} - x}{x - 2}$.
6. Find $\displaystyle\lim_{x\to 1} \frac{x^2 + x - 2}{(x+2)^2 - 9}$.
7. Find $\displaystyle\lim_{x\to 1} \frac{3x^4 - 5x^2 - 7x^3 + 11}{5x^5 - 3x^3 + 1}$.
8. Find $\displaystyle\lim_{x\to \frac{\pi}{4}} \frac{\cos(2x)}{\cos x - \sin x}$.
9. Find $\lim\limits_{t\to 0} \sin t \sin \frac{1}{t}$.
10. Find $\displaystyle\lim_{y\to 0} \frac{y^2}{2 + \sin(y^{137})}$.