One-sided limits

In the definition of limits we look at what happens as $x \to c$, but there are two ways that x can approach c. Namely we can approach it from below (i.e., $x < c$) or we can approach it from above (i.e., $x > c$). Sometimes it is convenient to limit ourselves to one direction when evaluating the limit, and other times it might be possible to only approach from one direction, which leads to *one-sided limits*.

$$\lim_{x\to c-} g(x) \quad \leftrightarrow \quad \text{limit as we approach } c \text{ from below}$$
$$\lim_{x\to c+} g(x) \quad \leftrightarrow \quad \text{limit as we approach } c \text{ from above}$$

These are also known respectively as the left limit and the right limit. (Beyonce implicitly mentions one-sided limits in her song "Irreplaceable" when she sings "to the left, to the left…".)

As an example we have

$$\lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-x}{x} = \lim_{x\to 0-} (-1) = -1; \text{ and}$$

$$\lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = \lim_{x\to 0+} 1 = 1.$$

Note that by limiting ourselves to one side we can simplify expressions, e.g., dropping the absolute value signs. When the left and right limits do not agree then the limit does not exist, conversely if both one-sided limits exist and agree then the limit exists. In general, when dealing with piece-wise functions (such as $|x|$) it is convenient to use one-sided limits to determine what happens at the glue point. Note some limits are easier handled when treated as a combination of one-sided limits.

The limit of $(\sin\theta)/\theta$ as $\theta \to 0$

A limit which plays an important role is

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1,$$

where θ is measured in radians. This is done by noting that for $0 < \theta < \frac{1}{2}\pi$ that

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

then using the squeezing theorem. This handles the limit from above, then by symmetry (function is even) the limit exists and equals 1.

Note that the limit is nice and clean for radians; if θ is in degrees then the limit goes to $\pi/180$. This is the reason Calculus people work in radians (1 is much easier to work with)!

Continuous functions

Closely related to the idea of limits is the idea of a continuous function. A function is continuous if it has "no breaks". Another way to say it is the function is continuous at $x = c$ if what we expect to happen at $x = c$ is what actually does happen, i.e.,

$$\lim_{x\to c} f(x) = f(c).$$

In particular, three things need to happen to be continuous: (1) $f(c)$ must be defined; (2) the limit must exist; (3) the preceding two have to agree. There are several types of discontinuities.

- *Removable discontinuity* The limit exists but either the function is not defined or the value of the function does not match the limit. The name removable comes from the idea that we can redefine the function at the point and we would no longer have a discontinuity.
- *Jump discontinuity* The left and right hand limits exist but are not equal.
- *Infinite discontinuity* The left, right, or both limits are $\pm\infty$.
- *Miscellaneous* Another possibility, not named, is that the limit does not exist (for example $\sin(1/x)$ at $x = 0$).

Examples of continuous functions include polynomials, x^a (in its domain), $\sin x$, $\cos x$, $\tan x$ (away from the vertical asymptotes), $\ln x$ and e^x. These form the building blocks of continuous functions and then we might ask for how can we combine continuous functions together. In particular we have that if $f(x)$ and $g(x)$ are continuous then so are $f(x) + g(x)$, $kf(x)$, $f(x)g(x)$, $f(x)/g(x)$ (when $g(x) \neq 0$) and $f(g(x))$ (the composite function also denoted as $(f \circ g)(x)$).

All of the functions that we will encounter are built up using the basic building blocks and rules for combining. The only other thing that might happen is that we might have a function which is defined piecewise. When this is the case the interesting question usually occurs at when two pieces meet.

The nice thing about a limit involving a continuous function is that we can plug in the point we are taking the limit to into the function. If we get a number out then we are done. If we get $0/0$ or other inconclusive forms (i.e., $\infty - \infty$, etc.), then that means we need to work on the limit some more using our various techniques.

Intermediate value theorem

The intermediate value theorem states that for a function $f(x)$ continuous on the interval $[a, b]$ that as the input ranges from $x = a$ to $x = b$ that the output will include every possible value between $f(a)$ and $f(b)$ (possibly more). In particular there is no possible jump in the output of a continuous function.

This can be used in finding roots, i.e., values where $f(x) = 0$. If we know $f(a) > 0$ and $f(b) < 0$ and f is continuous in an interval containing a and b then there *must* be at least one (possibly several) point between a and b which is a root. Successively cutting intervals that contain a 0 in half we can quickly find approximations to roots. We will see other methods to find roots of functions later in the course.

Infinite limits

Occasionally our limiting behavior involves infinity (∞). Infinity is not itself a number but rather an indication of what happens when we grow in an unbounded way. Limits involve infinity in two ways, namely we can look at what happens when the input gets arbitrarily large (i.e., $x \to \pm\infty$), or what happens when the function grows arbitrarily large (i.e., $f(x) \to \pm\infty$).

A function has a *horizontal asymptote* as $x \to \infty$ if $\lim_{x\to\infty} f(x) = L$ where L is finite (similarly for $x \to -\infty$). A functions has a *vertical asymptote* if $\lim_{x\to a} f(x) = \infty$ (or $-\infty$); in the latter case we often deal with one sided limits as the two different sides can have quite different behaviors.

To evaluate limits going to infinity it helps to divide out by the part which is growing fastest an see what happens.

Manipulating names

An important technique in limits (and in mathematics in general) is the ability to modify our limits by changing how we represent the limit. In essence we are *substituting* in another name for our limit, with the goal of being to result in an easier integral.

As an example, consider $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$. We can start by letting $u = \frac{1}{x}$, or equivalently by letting $x = \frac{1}{u}$. We have to now replace *every* occurrence of x and these are in two places, the function we are taking the limit of, and the limiting point. The new function becomes $\frac{\sin(u)}{u}$ (nice!); the new limiting point becomes 0 approached from above (i.e., as $x \to \infty$ then $u = \frac{1}{x} \to 0+$). We can conclude

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{u\to 0+} \frac{\sin(u)}{u} = 1.$$

Quiz 3 problem bank

1. Find $\lim_{\theta\to 0} \dfrac{\sin(4\theta) + \sin(5\theta)}{\sin(\theta) + \sin(2\theta)}$.
2. Find $\lim_{x\to 0} \dfrac{\sqrt{x+1} - 1}{\sin x}$.
3. Find $\lim_{t\to 0} \left(\dfrac{1}{3t} - \dfrac{1}{t(t+3)}\right)$.
4. Let $f(x)$ be piece-wise defined by
$$f(x) = \begin{cases} 5x+3 & \text{if } x \leqslant 1, \\ 3x-5 & \text{if } 1 < x. \end{cases}$$
Determine $\lim_{x\to 0} f(1+x^2)$.
5. Let $g(x)$ be piece-wise defined by
$$g(x) = \begin{cases} 1-x^2 & \text{if } x < 0, \\ 2+\frac{1}{2}x & \text{if } 0 \leqslant x \leqslant 2 \\ \frac{3}{16}x^3 & \text{if } 2 < x. \end{cases}$$
Determine $\lim_{x\to 0} g(x)g(x+2)$.
6. Determine a and b so that the following piece-wise function is continuous everywhere.
$$h(x) = \begin{cases} 6+x & \text{if } x < -1 \\ ax^2+bx & \text{if } -1 \leqslant x \leqslant 1 \\ 5-6x & \text{if } 1 < x \end{cases}$$
7. Find $\lim_{x\to 9} \dfrac{3-\sqrt{x}}{27-\sqrt{x^3}}$.
8. Find $\lim_{x\to\infty} \left(\sqrt{x^2+5x} - \sqrt{x^2-3x}\right)$.
9. Find $\lim_{x\to-\infty} \dfrac{6x+4}{\sqrt{4x^2-x-5}}$.
10. Given that $\lim_{x\to 1} g(x)$ exists and
$$\lim_{x\to 1}\left(\frac{1}{g(x)-5} - \frac{1}{g(x)+3}\right) = \frac{1}{6},$$
determine the possible value(s) of $\lim_{x\to 1} g(x)$.