Derivatives of trigonometric functions

Using the limit of $\sin(h)/h$ as $h \to 0$ and some basic trigonometric identities we are able to find derivatives for the trigonometric functions.

$$\frac{d}{dx}\big(\sin x\big) = \cos x$$
$$\frac{d}{dx}\big(\cos x\big) = -\sin x$$
$$\frac{d}{dx}\big(\tan x\big) = \sec^2 x$$
$$\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$$
$$\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$$
$$\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$$

(The last two of these are less commonly used.)

The important thing to remember is to get the signs on the derivative of sine and cosine correct. Since we can now be tested on these derivatives it is also helpful to know some of the values of the trigonometric functions.

θ	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0
$\cos\theta$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	DNE	0

Chain rule

The chain rule deals with the situation of how to take the derivative of a function inside of another function. Often a hint that we will use the chain rule is parentheses "(" and ")" or if when reading it out loud we use the word "of". For example $\sin(x^2)$ (read "sine of x squared") is the function x^2 inside the sine function.

$$\boxed{\frac{d}{dx}\big(f\big(g(x)\big)\big) = f'\big(g(x)\big)g'(x)}$$

We can of course go several layers deep, i.e., a function within a function within another function. In this way math is like onions, there are lots of layers and occasionally a few tears.

Common mistakes when using the chain rule include forgetting to multiply by $g'(x)$ or putting $f'(g'(x))$. So for example

$$\frac{d}{dx}\big(\sin(x^2)\big) = \cos(x^2)\cdot 2x$$

and not $\cos(2x)$ or $\cos(x^2)$. Also make sure you have clearly identified what the inside and outside functions are, i.e., $\sin(x^2)$ is a very different function than $(\sin x)^2$.

An important application of the chain rule is the power rule, which is the special case when $f(x) = x^a$.

$$\frac{d}{dx}\Big(\big(g(x)\big)^a\Big) = a\big(g(x)\big)^{a-1}g'(x).$$

Implicit differentiation

An explicit function is when you have

$$y = \boxed{\text{ stuff with } x\ }.$$

An *implicit* function is when we are given a relationship involving x and y but we do not (or can not) write y as a simple function of x. For example,

$$yx^2 + 3x\cos y = \sin x + y$$

gives a relationship between x and y, and for a given value of x we can solve for what possible value(s) y can be. So even though y is not given as a function of x we still can think of it as depending on x (being the case that we don't know exactly which function it is).

So to take the derivative we start with the equation defining the implicit relationship and take the derivative of both sides with respect to x. There are a few things to observe

1. If we take the derivative of two functions which are equal the derivatives are still equal.

2. We treat y as a function of x, in particular when it comes to expressions involving y we use the chain rule to take the derivative and when we finally get to the point where we need to write down the derivative of y we put dy/dx or y' (both mean the derivative of y, how easy is that!).

3. Finally we solve for dy/dx or y' by simply rearranging the result.

We can also find higher order derivatives by repeating this procedure.

So as an example if we differentiate both sides of the implicit relationship given above we have

$$x^2y' + 2xy + 3\cos y - 3x(\sin y)y' = \cos x + y'$$

which we then put terms with y' on one side and terms without y' on the other side and then factor out y' and divide by its coefficient. In this case we end up with

$$y' = \frac{\cos x - 2xy - 3\cos y}{x^2 - 3x\sin y - 1}.$$

And we can use this for example to see at $(0,0)$ that the implicitly defined curve has slope 3, i.e.,

$$y'\Big|_{(0,0)} = \frac{1-0-3}{0-0-1} = 2.$$

(The "$\big|_{(a,b)}$" is notation that stands for evaluate at the point (a, b).)

One easy mistake is to forget the y' terms in implicit differentiation, or forgetting to differentiate both sides (particularly common when one side is a constant!).

Quiz 6 problem bank

1. Find the tangent line to $y = \sin(\pi x^2) - \pi x$ at $x = 1$.

2. Given $f(t) = \sin t \cos t$, find $f^{(101)}(t)$.

3. Determine a and b so that $f(x)$ will have a derivative at $x = 0$, where $f(x)$ is defined piecewise by

$$f(x) = \begin{cases} 3\tan x - 2\sec x & \text{if } x < 0, \\ ae^{2x} + bx & \text{if } x \geqslant 0. \end{cases}$$

4. Consider the functions $f(x) = 2\tan x + 2\sec x$ and $g(x) = (\tan x + \sec x)^2 + 1$. Is $g(x)$ the derivative of $f(x)$ or is $f(x)$ the derivative of $g(x)$? Justify your answer.

5. Find $\dfrac{d}{dx}\Big(\sin\big(e^x + \cos\big(x^2 + x\tan x\big)\big)\Big)$

6. Given that $y = 4x + 1$ is tangent to $y = f(x)$ at $x = 1$, and that $g(x) = 3f(x^2) - x^3$, then determine *two* tangent lines to $y = g(x)$.

7. Let f be a twice differentiable function (i.e., has a second derivative) which satisfies the following:

$$\begin{array}{ccc} f(2) = -1 & f(3) = 5 & f(4) = 2 \\ f'(2) = 3 & f'(3) = 2 & f'(4) = -3 \\ f''(2) = 2 & f''(3) = 1 & f''(4) = 4 \end{array}$$

   Determine $g''(1)$ where $g(x) = x^2 f(x^2 + 2x)$.

8. An implicit function has a *vertical* tangent line if $dx/dy = 0$. Find the *four* points on the implicitly defined function $2y^3 + 3y^2 = 4x^2 + 5x + 1$ where we have a vertical tangent line.

9. Find the tangent line to the implicitly defined curve $y^3 + 3x^2y + x^3 = 5$ at $(1, 1)$.

10. Find $\dfrac{d^2y}{dx^2}$ given that $y + \frac{1}{3}y^3 = x + 137$.