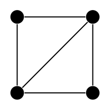
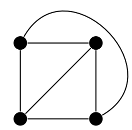
MATH314 HW 11

due **Apr 21** before class, **answer without justification will receive 0 points**. The typing the HW in LaTeX is optional.

1: A graph G is **outerplanar** if it can be drawn in the plane such that all vertices are incident with one face. Here are two examples. Cycle with additional edge is outerplanar, since all vertices are incident to the exterior face. On the other hand, K_4 is not outerplanar.



outerplanar



not outerplanar

Show that every outerplanar graph contains a vertex of degree at most 2.
Hint: If a leaf is there, you are done. So all vertices have degree at least 2. Following idea the thing works for cycle. Notice that it works for cycle even if you say that two of the vertices of the cycle cannot be used. Try to do just that - consider a cut by 2-vertices that are adjacent by an edge that is not in the outer and try to take a component that has no such cut...

Solution:
Let G be an outerplanar graph with n vertices and m edges. Add a new vertex v connected to every vertex of G. The resulting graph is planar, has $n+1$ vertices and $m+n$ edges. Now since the graph is planar, it satisfies

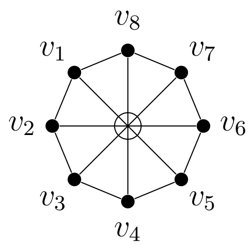
$$m + n \leq 3(n+1) - 6$$

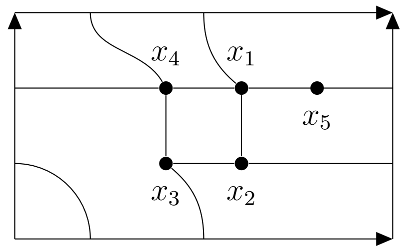
This translates to $m \leq 2n - 3$. So if every vertex has degree.

2: A maximal outerplanar graph is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph. Let G be a maximal outerplanar graph with at least three vertices. Prove that G is 2-connected.
Hint: Suppose that G has a cut vertex. And show that if it has a cutvertex, you may add one more edge that will violate the maximiality.

3: We define the **girth** of a 2-connected graph G to be the length of a shortest cycle in G. Prove that every 2-connected n-vertex planar graph of girth at least g, $g \geq 3$, has at most $g(n-2)/(g-2)$ edges.

4: Enumerate faces of the following graphs. The first graph is embedded in the plane with one cross-cap and the second graph is embedded in Torus.



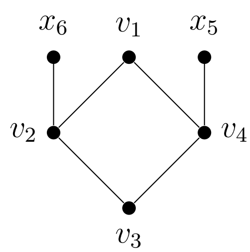


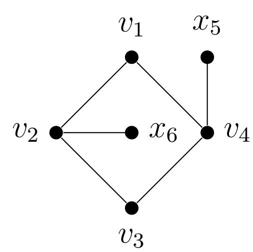
Mark the faces by either drawing several pictures and shading one face at a time or color the faces.

5: Find embedding of K_7 on torus.

6: Find two non-isomorphic embeddings of K_5 in
a) the Klein Bottle
b) torus.
So you should have 2 drawings in the Klein Bottle and two drawings in the Torus.

Non-isomorphic embeddings means that the set of faces are different. See the following example, where a graph is twice embedded in the plane such that the set of faces is different.





The left embedding has a face whose boundary is cycle $v_1v_2v_3v_4$ but there is no such face in the embedding on the right.