Chapters 1.1 and 1.2 - Introduction

A **simple graph** G is an ordered pair (V, E) of **vertices** V and **edges** E, where $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$.

$|V|$ is **order** of G

$|E|$ is **size** of G

Vertices of G are denoted by $V(G)$ and edges of G by $E(G)$.

If $\{u, v\} \in E$, then u and v are **adjacent** and called **neighbors**.

If $u \in V$ and $e \in E$ satisfy $v \in e$, then v and e are **incident**.

$\{u, v\}$ can be simplified to uv.

Edges are **adjacent** if they share vertices.

Drawing of G *assigns point* to V and *curves* to E, where endpoints of uv are u and v.

If $V(G) = \emptyset$ then G is a *null* graph.

Graph H is **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, notation $H \subseteq G$.

H is a **proper subgraph** if $H \subseteq G$, $H \neq G$ (and H is not a null graph).

H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$

H is a **induced subgraph** if $H \subseteq G$ and $\forall u, v \in V(H), uv \in E(G) \Rightarrow uv \in E(H)$.

If $X \in V(G)$, then $G[X]$ denotes induced subgraph H of G where $V(H) = X$.

We use $+$ and $-$ to denote adding edges or vertices to graph.

Walk in a graph G is a sequence $v_1, e_1, v_2, e_2, v_3, \ldots, v_n$, where $v_i \in V(G)$ and $e_i \in E(G)$, where consecutive entries are incident.

Trail is a walk without repeated edges.

Path is a walk without repeated vertices.

If walk,trail,path starts with u and ends with v, it is called $u - v$ walk,trail,path.

Length of a walk,trail,path is the number of edges.

Theorem 1.6 If a graph G contains $u - v$ walk, it also contains $u - v$ path.

Distance of u and v is the length of a shortest $u - v$ path, denoted $d(u, v)$.

Diameter of G, denoted by $diam(G)$ is maximum of $d(u, v)$ over all $u, v \in V$.

A $u - v$ path of length $d(u, v)$ is called **geodesic**.

Walk/Trail is *closed* if it is $u - u$ walk/trail. Otherwise it is *open*.

Circuit is a closed trail.

Closed trail with no repetition of vertices (except first and last) is **cycle**.

Graph is **conneted** if for all $u, v \in V$ exists $u - v$ walk.

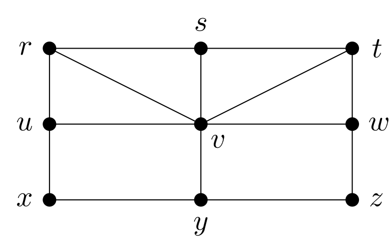
If graph in not connected, it is **disconneted**.

Connected **component** of G is a connected subgraph of G that is not a proper subgraph of any other connected subgraph of G.

Graph G is a **union** of graph $G_1, \ldots, G_k$ if G can be partitioned into $G_1, \ldots, G_k$. Notation $G = G_1 \cup G_2 \cup \cdots \cup G_k$.

1: *1.3* Let $S = \{2, 3, 4, 7, 11, 13\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G)$ for all $i, j \in S$ if $i + j \in S$ or $|i - j| \in S$. What is $|E(G)|$ and $|V(G)|$? What is diameter of G?

2: For the depicted graph G, give an example of each of the following or explain why no such example exists.



(a) An $x - y$ walk of length 6.
(b) A $v - w$ trail that is not a $v - w$ path.
(c) An $r - z$ path of length 2.
(d) An $x - z$ path of length 3.
(e) An $x - t$ path of length $d(x, t)$.
(f) A circuit of length 10.
(g) A cycle of length 8.
(h) A geodesic whose length is $diam(G)$.

3: *Theorem 1.7* Let R be the relation defined on the vertex set of a graph G by $u\ R\ v$, where $u, v \in V(G)$, if u is connected to v, that is, if G contains a $u - v$ path. Show that R is an equivalence relation. What are equivalence classes of R?

4: *Theorem 1.8* Let G be a graph of order 3 or more. If G contains two distinct vertices u and v such that $G - u$ and $G - v$ are connected, then G itself is connected.

5: *1.15* Draw all connected graphs of order 5 in which the distance between every two distinct vertices is odd. Explain why you know that you have drawn all such graphs.

6: *1.17* (a) Prove that if P and Q are two longest paths in a connected graph, then P and Q have at least one vertex in common.
(b) Prove or disprove: Let G be connected graph of diameter k. If P and Q are two geodesics of length k in G, then P and Q have at least one vertex in common.

Reading for next time: Chapters 1.1, 1.2, 1.3