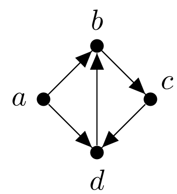
Chapter 4.3 Counting Spanning Trees Using Determinants

Let $\vec{G}$ be an oriented graph. Let matrix D be its *incidence matrix* defined as

$$d_{ik} = \begin{cases} -1 & \text{if } i \text{ is the tail of } \vec{e}_k \\ 1 & \text{if } i \text{ is the head of } \vec{e}_k \\ 0 & \text{otherwise} \end{cases}$$

1: Find the incidence matrix D for the following oriented graph



Let G be a graph on n vertices and $n-1$ edges. Let $\vec{G}$ be any orientation of G and let D be the incidence matrix of $\vec{G}$. Let $\overline{D}$ be obtained from D by deleting the first row corresponding to vertex x. Note that $\overline{D}$ is $(n-1) \times (n-1)$ matrix.

Lemma 1 $\det \overline{D} \in \{-1, 0, 1\}$. Moreover $\det \overline{D} = 0$ iff G is not a tree.

2: Show that if G of order at least 3 has a leaf $v \neq x$ then $|\det \overline{D}| = |\det \overline{D'}|$, where D' corresponds to $\vec{G} - v$.

3: Show that if G is a tree then $\det \overline{D} \in \{-1, 1\}$.

4: Show that if x is an isolated vertex then $\det \overline{D} = 0$.

5: Show that if $v \neq x$ is an isolated vertex then $\det \overline{D} = 0$.

6: Show that if $deg(v) \geq 2$ for all $v \neq x$ and $deg(x) \geq 1$ then G has too many edges.

7: Notice this finishes the proof of Lemma 1.

Crazy idea of computing the number of spanning trees of a general graph G. Take the incidence matrix of $\vec{G}$ and take all subsets of edges of size $n-1$ and compute absolute values of determinants of corresponding $\overline{D}$ and sum them up.

Theorem Binet-Cauchy Let A be an arbitrary matrix with $n-1$ rows and m columns. Then

$$det(AA^T) = \sum_I det(A_I)^2$$

where the sum is over all $(n-1)$-element subsets of $\{1, 2 \ldots, m\}$; that is $I \in \binom{\{1,2\ldots,m\}}{n-1}$ or also $I \subseteq \{1, 2 \ldots, m\}$ and $|I| = n-1$; and A_I is obtained from A by deleting all columns whose index is not in I.

8: Compute DD^T for D from question 1.

Laplace matrix of a graph G of order n is $n \times n$ matrix Q where

$$q_{ij} = \begin{cases} -1 & \text{if } ij \in E(G) \\ deg(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

9: Show that for any orientation of G holds $Q = DD^T$.

10: Let Q_{11} be obtained from Q by deleting the first row and the first column. Show that for any orientation of G holds $Q_{11} = \overline{D}\,\overline{D}^T$.

Matrix Tree Theorem Let G be a graph and Q its Laplace matrix. Then the number of spanning trees of G is equal to $\det(Q_{11})$ where Q_{11} is obtained from Q by deleting the first row and column.

11: Prove the Matrix Tree Theorem.

12: Count the number of spanning trees of the graph from question 1 using Matrix Tree Theorem.

13: Count the number of spanning trees of K_n using Matrix Tree Theorem.

14: Describe graphs that have exactly 3 spanning trees.

15: Show that if a graph G of order n is connected then it's Laplace matrix Q has rank $n-1$.

16: Use Cayley's formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees.

17: *Tougher* Count the number of spanning trees of $K_{m,n}$.

18: *Tougher,unknown* Prove or disprove: Let G be a graph with the minimum vertex degree at least 2; that is, $\delta(G) \geq 2$. Then there exists a spanning tree T of G such that for every vertex v in T that is adjacent to a leaf in T the following holds if $\deg_G(v) \geq 3$, then $\deg_T(v) \geq 3$.