Chapter 5.1 and 5.2 Connectivity

Edge e in a graph G is a **bridge** if $G - e$ has more connected components than G.

Graph G is **connected** if there exists $u - v$-walk for any two vertices of G.

How far is G from begin disconnected (how much connected?)?

A graph is k-**vertex connected** if G is K_{k+1} or for any $X \subset V(G)$ with $|X| = k - 1$ the graph $G - X$ is connected. (one needs to remove at least k vertices to disconnect G)

When is graph 1-connected?

Vertex v in a graph G is a **cut-vertex** if $G - v$ has more connected components than G.

1: Find graph that has more cut-vertices than bridges and graph that has more bridges than cut-vertices.

2: Let v be a vertex incident with a bridge in a connected graph G. Then v is a cut-vertex of G if and only if $\deg v \geq 2$.

3: A vertex v of a connected graph G is a cut-vertex of G if and only if there exist vertices u and w distinct from v such that v lies on every $u - w$ path of G.

4: Show that every connected graph has at least 2 vertices that are not cut-vertices. (Consider u and v where $dist(u, v) = diam(G)$, distance of u and v is the diameter - the maximum possible one.)

Connected graph with no cut-vertices is called **non-separable**.

Theorem 5.7 A graph of order at least 3 is non-separable if and only if every two vertices lie on a common cycle.

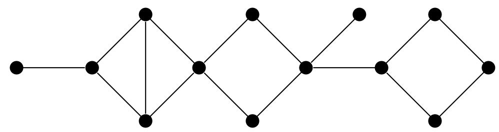
5: Show any two vertices on common a cycle $\Rightarrow$ no cut-vertex.

6: Show if G is non-separable of order at least 3 $\Rightarrow$ any two adjacent vertices are on a common cycle.

7: Show if G is non-separable of order at least 3 $\Rightarrow$ any two vertices are on a common cycle.

A maximal nonseparable subgraph of a graph G is called a **block** of G.

8: Find blocks in the following graph:



Theorem 5.8 Let R be the relation defined on the edge set of a nontrivial connected graph G by $e\ R\ f$, where $e, f \in E(G)$, if $e = f$ or e and f lie on a common cycle of G. Then R is an equivalence relation.

9: Prove Theorem 5.8. Symmetry and reflexivity is easy. Transitivity is the thing to prove.

10: Let B_1 and B_2 be distinct blocks in a nontrivial graph G. Show that B_1 and B_2 are edge-disjoint.

11: Let B_1 and B_2 be distinct blocks in a nontrivial graph G. Show that B_1 and B_2 have at most one vertex in common.

12: Let B_1 and B_2 be distinct blocks in a nontrivial graph G. Show that B_1 and B_2 have a vertex v in common, then v is a cut-vertex of G

13: *5.7* Prove that if T is a tree of order at least 3, then T contains a cut-vertex v such that every vertex adjacent to v, with at most one exception, is an end-vertex.

14: *5.11* Prove that if G is a graph of order $n \geq 3$ such that $\deg v \geq n/2$ for every vertex v of G, then G is nonseparable.

15: Let G be a graph. Let T be a graph whose vertices correspond to blocks in G and two vertices in T are adjacent if the corresponding blocks share a vertex. Show that T is a tree.

16: A **cactus** is a connected graph in which every block is an edge or a cycle. Prove that the maximum number of edges in a simple n-vertex cactus is $\lfloor 3(n-1)/2 \rfloor$. (Hint: $\lfloor x \rfloor + \lfloor y \rfloor$.)

17: *5.13* Prove or disprove: If G is a connected graph with cut-vertices and u and v are vertices of G such that $d(u,v) = diam(G)$, then no block of G contains both u and v.

For a graph G denote by $\kappa(G)$ the cardinality of minimum $X \subset V(G)$ such that $G - X$ is disconnected. Define $\kappa(K_n) = n - 1$. It is called **connectivity**.

18: What is $\kappa(G)$, where G is the Petersen's graph?

19: Show that for every graph G holds $\kappa(G) \leq \delta(G)$. Recall $\delta(G)$ is the minimum degree of G.

Paths are **internally disjoint** if they do not share any vertices of degree 2. They may share vertices of degree one (the end-vertices).

20: *Tougher* Prove Menger's theorem. Let G be a graph with $\kappa(G) \geq k$. Let u, v be any two distinct vertices in G. Show that there exist internally disjoint paths $P_1, P_2, \ldots, P_k$ where u and v are the endpoints. (Can you do at least $k = 2$?)

21: *Tougher* Find a graph that is maximizing the number of induced copies of C_5 and have no triangles.

Reading for next time - Chapter 5.3.