Chapter 5.4 Menger's Theorem

Paths P_1 and P_2 are **internally disjoint** if their intersection contains only endpoints.

Theorem 5.16 (Menger's Theorem) Let u and v be nonadjacent vertices in a graph G. The minimum number of vertices in a $u - v$ separating set equals the maximum number of internally disjoint $u - v$ paths in G.

Proof. Let the minimum separating set be U. Use induction on $|U| = k$. Then use induction on the number of vertices and edges.

1: Case 1: vertices u and v have a common neighbor $x \in U$.

2: Case 2: There exists vertex in U not adjacent to u and a vertex not adjacent to v.

3: Case 3: Every U has all vertices adjacent to u and none to v or vice versa.

Theorem 5.17 A nontrivial graph G is k-connected for some integer $k \geq 2$ if and only if for each pair u, v of distinct vertices of G there are at least k internally disjoint $u - v$ paths in G.

4: Prove theorem 5.17 for complete graphs.

5: Show $\Leftarrow$ direction.

6: Show $\Rightarrow$ direction if u and v are not adjacent.

7: Show $\Rightarrow$ direction if u and v are adjacent.

8: Let G be a k-connected graph and let S be any set of k vertices. Show that if a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.

9: Show that if G is a k-connected graph and u, $v_1, v_2, \ldots, v_k$ are $k+1$ distinct vertices of G, then there exist internally disjoint $u - v_i$ paths ($1 \leq i \leq k$) in G.

Theorem 5.20 If G is a k-connected graph, $k \geq 2$, then every k vertices of G lie on a common cycle of G.

We prove Theorem 5.20 by growing a cycle. Let $S = \{v_1, \ldots, v_k\}$. Since $k \geq 2$, there is a cycle containing v_1 and v_2. Let C be a cycle containing vertices $\{v_1, \ldots, v_l\}$. We will use the previous question to extend the cycle.

10: Show that if C is a cycle of length l formed by vertices $\{v_1, \ldots, v_l\}$, then there exists a cycle containing vertices $\{v_1, \ldots, v_l, v_{l+1}\}$.

11: Show that if C is a cycle of length $> l$ containing vertices $\{v_1, \ldots, v_l\}$, then there exists a cycle containing vertices $\{v_1, \ldots, v_l, v_{l+1}\}$.

12: *5.33* Let G be a 5-connected graph and let u, v and w be three distinct vertices of G. Prove that G contains two cycles C and C' that have only u and v in common but neither of which contains w.

Harary graph $H_{r,n}$ is a graph on n vertices $v_1, \ldots, v_n$ that form a cycle C defined as follows. If $r = 2k$ is event, then $H_{r,n} = C^k$ (recall that we take power of cycle). If $r = 2k+1$ is odd and $n = 2l$ is even, then $H_{r,n}$ is obtained from C^k by adding edges $v_i v_{i+l}$, where $1 \leq i \leq l$. If $r = 2k+1$ is odd and $n = 2l$ is odd, then $H_{r,n}$ is obtained from C^k by adding edges $v_i v_{i+l+1}$, where $1 \leq i \leq l$ and edge $v_1 v_{1+l}$.

13: Draw $H_{2,6}$, $H_{3,8}$, $H_{3,9}$.

14: Show that for any two integers r, n with $2 \leq r < n$ holds $\kappa(H_{r,n}) = r$.

15: *Open* Prove or disprove that if G is a 3-connected graph, then no longest cycle in G is induced.

Reading for next time - Chapter 5.4.