Chapter 8.1 - Matchings I

1: As a result of doing well on an exam, six students Ashley (A), Bruce (B), Charles (C), Duane (D), Elke (E) and Faith (F) have earned the right to receive a complimentary textbook in either algebra (a), calculus (c), differential equations (d), geometry (g), history of mathematics (h), programming (p) or topology (t). There is only one book on each of these subjects. The preferences of the students are

A: d,h,t; B: g p t; C: a,g,h; D:h,p,t; E:a,c,d; F: c,d,p

Can each of the students receive a book he or she likes?

2: What if the preferences are

A: d,h,t; B: g p t; C: d,h,t; D:g,p,t; E:d,h,t; F: d,h,t

Let G be a graph. **Matching** M is a subset of edges of G such that every vertex of G is incident to *at most* one edge in M.

A matching M in G is **perfect** if every vertex of G is incident to *exactly* one edge in M.

Let G be a graph and $X \subseteq V(X)$. Define the **neighborhood** of X, denoted by $N(X)$, as

$$N(X) = \{v \in V(G) : \exists x \in X \text{ such that } vx \in E(G)\}.$$

Let G be a bipartite graph with bipartition U and W where $|U| \leq |W|$. The graph G satisfies **Hall's condition** is $|N(X)| \geq |X|$ for every $X \subseteq U$.

Theorem 8.3 Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$. Then G contains a matching of cardinality r if and only if G satisfies Hall?s condition.

3: Show $\Rightarrow$ of Theorem 8.3. Show that if G contains a matching of cardinality r, then G satisfies the Hall's condition.

We prove $\Leftarrow$ by induction on $|U|$.

4: Show the base case of the induction for $r = 1$.

Now we consider two cases based on $|N(S)|$, where $S \subset U$.

5: Case 1: Assume that *every* proper subset S of U satisfies $|N(S)| > |S|$. Find the desired matching. (recall that proper subset means $1 \leq |S| < |U|$.)

6: Case 2: There exists a proper subset X of U such that $|N(X)| = |X|$. Find the desired matching.

A set of edges is **independent** it if is a matching. (two names for the same thing)

Edge independence number $\alpha'(G)$ of a graph G is the size of the largest matching in G.

7: Determine $\alpha'(C_6)$, $\alpha'(C_7)$, $\alpha'(K_5)$. Conclude in general what is $\alpha'(C_n)$, $\alpha'(K_n)$. Let H be a K_4, where every edge is once subdivided. Determine $\alpha'(H)$.

A vertex and incident edge **cover** each other.

An **edge cover** of a graph G is a set of edges such that every vertex is covered. (Note G should have no isolated vertices.)

The **edge covering number** $\beta'(G)$ of a graph G is the smallest cardinality of an edge cover.

8: Determine $\beta'(C_6)$, $\beta'(C_7)$, $\beta'(K_5)$. Conclude in general what is $\beta'(C_n)$, $\beta'(K_n)$. Let H be a K_4, where every edge is once subdivided. Determine $\beta'(H)$.

Theorem 8.7 For every graph G of order n containing no isolated vertices,

$$\alpha'(G) + \beta'(G) = n.$$

9: Start with maximum matching, create a cover from it and show $\alpha'(G) + \beta'(G) \leq n$.

10: Start with minimum edge-cover C, study how the graph induced by C looks like (can it have triangles? Long paths?) and construct a matching from it to show that $\alpha'(G) + \beta'(G) \geq n$. (hint: count in forests)

11: Show that every r-regular bipartite graph ($r \geq 1$) has a perfect matching.

12: Show that every tree has at most one perfect matching.

13: Give an example of a connected non-Hamiltonian graph that contains two disjoint perfect matchings. (A graph on n vertices is Hamiltonian if it contains a cycle of length n.

14: Show that the Petersen graph does not contain two disjoint perfect matchings. (Try by contradiction and take union of the matchings.)

15: *Open Problem* Let G be an $(a + b + 2)$-edge-connected graph. Does there exist a partition $\{A, B\}$ of $E(G)$ so that (V, A) is a-edge-connected and (V, B) is b-edge-connected?